A TEXTBOOK OF
PURE MATHEMATICS

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PREFACE TO THE SECOND EDITION

The first edition of this book met with a very wide appreciation, and as a consequence thereof, the present edition should have been brought out much earlier. The authors and the Publishers regret very much the delay that has been caused mainly on account of abnormal conditions created by the War. While all the essential features of the previous edition have been retained, the book has been revised thoroughly and substantial additions have been made in the subject matter. The logical arrangement of having 'Complex Numbers' in the Introductory Chapter proved too difficult for the beginners, and it has consequently been considered advisable to shift 'Complex Numbers' to Chapter IX, in a very much improved form. Several other portions, included mainly in Chapters VI-XIII, have also been re-written and enlarged. Statements of Theorems have been made more precise and proofs more rigorous. This should not be considered tantamount to making the subject difficult, as the treatment has at the same time been made more explanatory. All this accounts for an increase of more than fifty pages in the volume of the book.

Quite a large number of questions selected from recent Examination Papers have been added. In the case of the more difficult among them, either full solutions or hints have been given, partly with a view to give the student necessary guidance, and partly in order to excite his interest in the work. The student is, however, strongly advised to exercise great restraint in referring to the hints or the solutions. He should invariably start with an independent attempt to get at the solution.

It is hoped that in the present form the book will meet the requirements of the B.A., and B.Sc., Pass Course and Honours students of Indian Universities, even to a greater extent than did the previous edition.

The thanks of the authors are due to Prof. A. N. Ganguli who has read through the mss. of Chapters VI and IX. The Publishers and Printers are also to be thanked for their whole-hearted co-operation in seeing the book through the press.

LAHORE:
February, 1946.

G. L. GUPTA
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EXTRACTS FROM THE PREFACE TO THE IST EDITION

The recent changes made by the Punjab University in the Syllabus for B. Course of Mathematics created a demand for a revision or amplification of the book entitled "*A Course in Higher Algebra*" by the authors. After much careful thought it was decided to bring out a book which would fully meet the requirements of Paper (*a*) in B Course, which now consists of Algebra (one-third of the paper), Theory of Equations (one-third), and Trigonometry (one-third). The necessities, therefore, of giving a logical treatment of the four fundamental operations on complex numbers required that the introductory chapter should be on "Real and Complex Numbers". That suggested the present title of the book which, however, does not pretend to be an introductory treatise on all topics pertaining to Pure Mathematics. It is, however, hoped that treatment of the topics included will be found to be free from logical flaws and such as to lay, within obvious limits, correct foundations for a further pursuit of the subject.

With our "*A Course in Higher Algebra*" as nucleus the present book has, of course, been written afresh. All that seemed to be unnecessary in the old book has been omitted and the rest of the matter has been so compressed as to give the book with all necessary additions a reasonable size. * * *

Some extracts from the preface to our old book, are equally applicable to the present book and are reproduced below :—

We have tried to treat the subject in as easy a manner as could be possible. In the Theory of Equations, we have omitted the Theorems of Fourier and Sturm and have only given a statement of Budan's Theorem. The roots of equations, unless they are nearly equal, can always be separated by other methods which involve less labour ; and as any elaborate discussion of equations, with nearly equal roots, is outside the scope of the book, the necessity for the above Theorems does not arise. The treatment of determinants will be found to be sufficiently rigorous and is based on the Theorem of Permutations. In the chapter on Infinite Series, we recognize the restrictions to which the train of reasoning is subject. The theory of limits, without which the first facts relating to convergence cannot have a firm foundation, are beyond the scope of this book. The treatment of the subject, as presented here, is not, therefore, free from assumptions and limitations. But we may say that the subject is treated in such a manner, that the student will have nothing

to unlearn, should he choose afterwards to proceed to a rigorous study of Infinite Series. * * *

The book contains a very large number of examples. Some have been constructed for this book, and others collected from many sources. Apart from giving solutions of examples illustrative of the book-work, we have given solutions or hints for difficult examples, whenever we have found it necessary, and we hope these will encourage the student to tackle successfully even the more difficult examples that may be found at the end of the chapters.

* * * The writers on the subject to whom we are indebted are too numerous to be mentioned here. Their names will be found in the bibliography given at the end of the book. That will also indicate to the student the sources upon which he could draw in case he felt more deeply interested in the subject.

Considering the books mostly in use in Indian Universities for Pass Course, the present book, perhaps, strikes a new line. We shall feel grateful to the teachers using the book for the valuable criticism or suggestions they may have to offer for its improvement.

LAHORE :
1st. Jan. 1942.

G. L. GUPTA
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A TEXT-BOOK OF
PURE MATHEMATICS

CHAPTER I

REAL NUMBERS. ELEMENTARY PROPERTIES OF EQUATIONS

1·1. The question as to the '*nature* and *meaning* of numbers' is one which concerns the philosophers. The number concept is considered by mathematicians as entirely independent of notions of time and space, and is an immediate result of the laws of thought. Numbers are free creation of the human mind. 'The whole of arithmetic is a natural consequence of a simple arithmetic act, that of counting, and counting itself is nothing else than the successive creation of the infinite series of positive integers in which each individual is defined by the one immediately preceding ; the simplest act is the passing from an already-formed individual to the consecutive new one to be formed.' Addition is the repeated application of the simplest act into one single act, and multiplication is the repeated application of addition. The individuals so formed are called **the natural numbers** which are denoted by

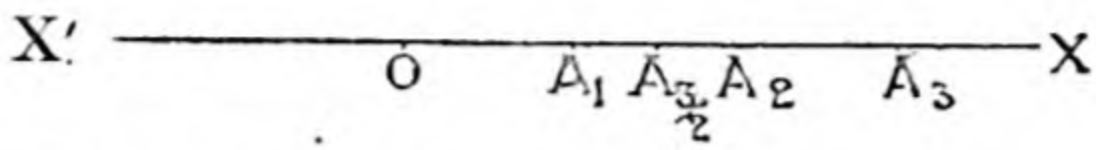
1, 2, 3, 4,...

While operations of addition, and multiplication always give a number of the system, the inverse operations of subtraction and division are not always possible in the *domain of positive integral numbers.* This limitation led to the creation of negative and fractional numbers by the human mind. The system of numbers so created is called the **rational number system** which we denote by **R.**

1·11. Representation and Properties of the System R.

The numbers of the system **R** can be represented by points of a straight line. Taking for granted that we understand what is meant by a *st. line* and a *segment of a line*, we take an arbitrary point O as the origin and another point A_1, such that the segment OA_1 is taken as unit. Any other number p/q can be represented by taking a length $OA_p = p$, and dividing it into

q equal parts. The first point of division $A_{p/q}$, such that $OA_{p/q}=p/q$, represents the number p/q. The point $A_{p/q}$ will be to the right or left of O, according as p/q is positive or negative.



This complete correspondence between the points of a line and rational numbers suggests an important property, that the *system of rational numbers is everywhere dense*. It means that, however close two rational numbers may be, it is possible to find a rational number between them.

1·12. Irrational Numbers. The four fundamental operations of arithmetic in the *rational domain* produce a member of the system. The system **R** is, however, insufficient to solve such equation as $x^2=2$. This needs the creation of new types of numbers distinct from the members of **R**.

First consider a terminating decimal, e.g.

$$\cdot a_1 a_2 a_3 \ldots a_n .$$

This can at once be reduced to the fraction of the form

$$\frac{a_1 a_2 a_3 \ldots a_n}{10^n},$$

which is a member of **R**.

In the second place, we can easily see that

$$\cdot\dot{9}=\frac{9}{10}+\frac{9}{10^2}+\frac{9}{10^3}+\ldots=1.$$

Finally consider a recurring decimal

$$\cdot a_1 a_2 \ldots a_n \dot{a}_1 a_2 \ldots \dot{a}_m$$

$$=\frac{a_1 a_2 \ldots a_n}{10^n}+\frac{a_1 a_2 \ldots a_m}{10^{n+m}}+\frac{a_1 a_2 \ldots a_m}{10^{n+2m}}+\ldots$$

$$=\frac{a_1 a_2 \ldots a_n}{10^n}+(a_1 a_2 \ldots a_m)\frac{\frac{1}{10^{n+m}}}{1-\frac{1}{10^m}}$$

$$=\frac{a_1 a_2 \ldots a_n}{10}+\frac{a_1 a_2 \ldots a_m}{10^n(10^m-1)}$$

$$=\frac{a_1 a_2 \ldots a_n a_1 a_2 \ldots a_m - a_1 a_2 \ldots a_n}{999 \ldots 900 \ldots 0},$$

where the denominator contains m 9's followed by n zeros. Thus in each case we get a number of the form p/q which is a member of **R**.

Conversely, it can be proved that every number p/q, where p and q are integers, can be expressed in one of the above forms. We may also note that every terminating decimal can be expressed as a recurring decimal whose recurring part consists entirely of 9's, *e.g.*

$$0{\cdot}315 = 0{\cdot}3149.$$

From the above discussion it follows that a decimal fraction which does not terminate or recur, cannot represent a rational number. Accordingly we define :—

Every infinite non-recurring decimal

$$g + {\cdot}a_1\, a_2\, a_3 \ldots$$

is a real irrational number.

The domain **R** of rational numbers, which is represented by terminating or recurring decimal fractions, together with the irrational numbers represented by infinite non-recurring decimals, constitutes the domain of real numbers, or the arithmetical continuum.

We can establish a complete (1, 1) correspondence between the arithmetical continuum and the points of a directed line (geometrical continuum) as in Art. 1·11.

Another method is to represent the number by a displacement along the line, the positive number x_1 being represented by a displacement of x units from left to right, and the negative number x_2 being represented by a displacement of x_2 units from right to left.

1·2. Functions, Polynomials. Let x and y be two variables, and E_1, E_2 their fields of variability. If a correspondence can be established between E_1 and E_2 by whatever law, we say, y is a function of x, and write $y = f(x)$, or $y = \phi(x)$, (say).

The variable x is called the independent variable, and y the dependent variable. It is obvious from the definition that the role of the independent variable x, and of the dependent variable y, can be interchanged, so that y can be taken as the independent variable and x the dependent variable.

Suppose n is a positive integer, and a_0, a_1, a_2...are some constants, and $a_0 \neq 0$, then the expression

$F(x) \equiv a_0x^n + a_1x^{n-1} + \ldots\ldots + a_{n-1}x + a_n$ is called a **polynomial of degree** n in the variable x. The relation

$$F(x) = 0$$

is called an **equation of degree, or order,** n.

For the values 1, 2, 3, 4, of n, the equation $F(x)=0$ is called a **linear, quadratic, cubic, biquadratic (or quartic), respectively.**

Any value of x for which the polynomial $F(x)$ vanishes, is called a **root** of the equation $F(x)=0$, or **zero** of $F(x)$.

The main object of **Theory of Equations** is to find the roots of the equation $F(x)=0$, *i.e.*, to solve the equation.

1·3. (Factor Theorem). *If h is a root of the equation $f(x)=0$, then the polynomial is exactly (i.e., without a remainder) divisible by $x-h$, and conversely.*

Let $f(x)$ be divided by $x-h$ in the ordinary way until a remainder R, if possible, is left which does not involve x. Let Q be the quotient. Then we have the identity

$$f(x) \equiv (x-h)\,Q + R.$$

Putting $x=h$, we have $f(h)=R$; but $f(h)=0$, since h is a root of the equation $f(x)=0$. It thus follows that $R=0$, *i.e.* $f(x)$ is exactly divisible by $x-h$.

To prove the *converse*, we notice that since $f(x)$ is exactly divisible by $x-h$, we have

$$f(x) \equiv (x-h)\,Q.$$

Putting $x=h$ in this identity, we get $f(h)=0$, which proves that h is a root of the equation $f(x)=0$.

2.1. We have seen that $f(h)=R$. If h be not a root, we conclude that the value of the polynomial for $x=h$, equals the remainder obtained by dividing $f(x)$ by $x-h$.

EXAMPLES I (*a*)

1. Find the values of a and b so that 3, 1 may be roots of the equation : $2x^4 - 7x^3 + ax + b = 0$.

2. Find a, b, c if $x^5 + 2x^3 + ax^2 + bx + c$ be divisible by x^3+1.

[**Hint.** Divide the expression by x^3+1 and equate to zero the coefficients of x^2, x and the absolute term of the remainder.]

3. Show that $(x-1)^{2n} - x^{2n} + 2x - 1$ is divisible by

$$2x^3 - 3x^2 + x.$$

Solution : $2x^3 - 3x^2 + x = x\,(2x-1)(x-1)$.

If $x=0$, $f(x) \equiv (x-1)^{2n} - x^{2n} + 2x - 1 = 0$, which shows that x is a factor of $f(x)$.

Similarly, $f(x)$ vanishes for $x=1$ and $x=\frac{1}{2}$.

Hence $f(x)$ is divisible by $2x(x-1)(x-\frac{1}{2})$, or $2x^3 - 3x^2 + x$.

4. For what values of n is the polynomial $x^{3n}-x^{2n}+x^n-1$ divisible by x^3-x^2+x-1.

5. If a be a real root of the cubic $x^3+px^2+qx+r=0$, of which the coefficients are real, then the other two roots are real, if $p^2-4q-2pa-3a^2 \geqslant 0$. (*Math. Trip.*)

Solution : Since a is a root, $x-a$ divides x^3+px^2+qx+r exactly. If we carry on the process of division, we get the quotient

$$x^2+(p+a)x+a^2+pa+q.$$

Now the equation $x^2+(p+a)x+a^2+pa+q=0$ will have real roots, if $(p+a)^2-4\,(a^2+pa+q) \geqslant 0$,
which gives the result.

SYNTHETIC DIVISION.

1·31. Computation of Polynomials. The labour of computing the value of a polynomial for a particular value of x can be saved by a simple device explained below.

Let the polynomial be $a_0x^3+a_1x^2+a_2x+a_3$, and $x=h$. For $x=h$, $a_0x^3=a_0x^2h$, so that the sum of the first two terms is $(a_0h+a_1)x^2$ and this equals $(a_0h+a_1)hx$, where $x=h$. Thus the sum of the first three terms is $(a_0h^2+a_1h+a_2)x$. Hence the value of the cubic is $a_0h^3+a_1h^2+a_2h+a_3$.

If we write the coefficients in a line, supplying the intermediate missing powers of x with zero coefficients, the work is neatly exhibited thus

h	a_0	a_1	a_2	a_3
		a_0h	$a_0h^2+a_1h$	$a_0h^3+a_1h^2+a_2h$
	a_0	a_0h+a_1	$a_0h^2+a_1h+a_2$	$a_0h^3+a_1h^2+a_2h+a_3$

It can be easily verified that the expressions a_0, a_0h+a_1, $a_0h^2+a_1h+a_2$ appear as coefficients of the quotient during the process of division of $f(x)$ by $x-h$, and $a_0h^3+a_1h^2+a_2h+a_3$ is the remainder. The method exhibited above is in *essence* a method of division of $f(x)$ by $x-h$. This process is called *synthetic division.*

For the sake of illustration. consider the following case :

Let $x=5$, $f(x)=x^4+5x^3-3x+7$.

5	1	5	0	−3	7
		5	50	250	1235
	1	10	50	247	1242

Thus the value of x^4+5x^3-3x+7 for $x=5$ is 1242, and the quotient obtained by dividing this expression by $x-5$ is $x^3+10x^2+50x+247$.

1·32. The Behaviour of a Polynomial at Infinity.

Suppose that the function $f(x)$ is given by

$$f(x)=a_0x^n+a_1x^{n-1}+\ldots\ldots+a_{n-1}x+a_n \qquad (1)$$

If $x>0$, each power of x is positive, and, therefore,

$$f(x) \leqslant a_0x^n+\mathrm{A}_1x^{n-1}+\mathrm{A}_2x^{n-2}+\ldots\ldots+\mathrm{A}_{n-1}x+\mathrm{A}_n,$$

and $$f(x) \geqslant a_0x^n-(\mathrm{A}_1x^{n-1}+\mathrm{A}_2x^{n-2}+\ldots\ldots+\mathrm{A}_{n-1}x+\mathrm{A}_n),$$

where $|\,a_1\,|=\mathrm{A}_1$, $|\,a_2\,|=\mathrm{A}_2$ etc.

Also when $x>1$, $x^n>x^{n-1}>x^{n-2}\ldots>x>1$, consequently

$$a_0x^n+(\mathrm{A}_1+\mathrm{A}_2+\ldots+\mathrm{A}_n)x^{n-1}>f(x)$$
$$>a_0x^n-(\mathrm{A}_1+\mathrm{A}_2+\ldots\ldots+\mathrm{A}_n)x^{n-1},$$

i.e.
$$\left(a_0+\frac{\mathrm{A}}{x}\right)x^n>f(x)>\left(a_0-\frac{\mathrm{A}}{x}\right)x^n,$$

where $\mathrm{A}=\mathrm{A}_1+\mathrm{A}_2\ldots\ldots+\mathrm{A}_n$.

Now choose $\varepsilon>0$, however small; then we can make $(\mathrm{A}/x)<\varepsilon$ by taking $x>\mathrm{A}/\varepsilon$. Thus for all values of $x>\mathrm{A}/\varepsilon$

$$(a_0+\varepsilon)x^n>f(x)>(a_0-\varepsilon)x^n,$$

i.e. the function $f(x)$ behaves like a_0x^n, for large positive values of x.

Similarly, *if x is negative, but numerically large, $f(x)$ behaves like a_0x^n, its leading term.*

EXAMPLES I (*b*)

1. Form the equations whose roots are (*i*) 4, -3, -5, 2; (*ii*) 1, $\frac{3}{5}$, $\frac{-7}{5}$, -5.

2. Find the equation whose roots are $\omega m+\omega^2 n$, $\omega^2 m+\omega n$, where $\omega^3=1$.

3. Find the equation whose roots are $m+n$, $\omega m+\omega^2 n$, $\omega^2 m+\omega n$, where $\omega^3=1$.

4. Find the quotient and the remainder when $x^4-5x^3+3x^2-7x+1$ is divided by $x+2$.

Here $h=-2$;

1	-5	3	-7	1	$(-2$
	-2	14	-34	82	
1	-7	17	-41	83	Remainder.

The quotient is $x^3-7x^2+17x-41$, and the remainder 83, (which is the value of the function when $x=-2$).

5. Find the value of $2x^5-x^3+2x-1$ for $x=-2$.

6. Given that 8 is the root of the equation

$$2x^3-31x^2+112x+64=0,$$

solve the equation completely.

7. Given that -4 is a root of the equation

$$x^4-2x^3-13x^2+38x-24=0,$$

solve it.

8. Depress the equation $x^4-2x^3-13x^2+38x-24=0$ by one dimension.

[Hint. 1 is seen to be a root by inspection.]

If another root of the equation is known to be -4. find the remaining roots.

9. Tabulate the values of the function

x^3-x^2+2x-3, when $x=2, -2, 3, -3$.

10. Divide $4x^5-10x^4+6x^3-7x^2+9x-11$ by $x-2$.

11. Divide $2x^4-5x^3+7x-5$ by $x+2$.

1·4. The Fundamental Theorem :—*Every equation of the form $f(x)=0$ has a root, where $f(x)$ is a polynomial in x.*

The proof of this theorem is beyond the range of the present work. We shall, therefore, *assume* it.

It follows that there exists at least one value of x for which the polynomial $f(x)$ has a given value.

For, if the given value be k, the equation $f(x)-k=0$ has at least one root.

1·41. *Every equation of the nth degree has n roots, and no more.*

Let the given equation be

$$f(x)\equiv p_0x^n+p_1x^{n-1}+p_2x^{n-2}\ldots\ldots+p_n=0.$$

The equation $f(x)=0$ has a root, real or imaginary*. Let it be h_1. Then $f(x)$ is divisible by $x-h_1$, so that,

$$f(x)\equiv(x-h_1)f_1(x),$$

where $f_1(x)$ is polynomial of $n-1th$ degree. Again, the equation $f_1(x)=0$ has a root, real or imaginary. Let it be h_2, so that $f_1(x)=(x-h_2)f_2(x)$, where $f_2(x)$ is a polynomial of $n-2th$ degree.

Hence $f(x)\equiv(x-h_1)(x-h_2)f_2(x)$.

Proceeding in this way, we obtain the identity

$$f(x)\equiv(x-h_1)(x-h_2)\ldots\ldots(x-h_n)f_n(x),$$

where $f_n(x)$ is of degree $(n-n)$, or zero, *i.e.* does not contain x. Comparing the coefficients of x^n on both sides, we find

$$f(x)\equiv p_0x^n+p_1x^{n-1}+\ldots+p_n\equiv p_0(x-h_1)(x-h_2)\ldots(x-h_n)\ldots(1).$$

*It is assumed that the student possesses elementary ideas about complex or imaginary numbers, and knows the four fundamental operations of addition, subtraction, multiplication and division of those numbers, these will be introduced formally in a subsequent chapter.

Obviously, $f(x)$ vanishes when x is equal to any one of the numbers h_1, $h_2 \ldots\ldots h_n$, and, therefore, the equation has n roots.

Also the equation cannot have more than n roots, for then the product of more than n linear factors of the form $x-h$ would be a factor of the polynomial, *i.e.* a polynomial of the nth degree has a factor of degree higher than n, which is impossible.

1·42. When some, or all, of the h's are equal in one or more groups, the number of distinct roots will be diminished. But the equation will still be said to possess n roots, the equal values giving the corresponding equal roots. In this way, the generality of the theorem is maintained. The right hand side of (1) will still contain n factors, some of the factors being repeated.

1·43. *If the equation* $f(x)=0$ *is satisfied by more than n values of* x, *then all the coefficients of* $f(x)$ *vanish*, i.e. $f(x)$ *is identically zero.* The proof of this is postponed. 4.

1·44. *If an equation with real coefficients, has a root* $\alpha+i\beta$, *it has also the root* $\alpha-i\beta$, *where* α *and* β *are real.*

Since $\alpha+i\beta$ is a root,

$$f(x)\equiv\{x-(\alpha+i\beta)\}\,Q.$$

where Q is a polynominal of degree $n-1$.

In this identity write $-i$ for i ; and since $f(x)$ does not contain i explicitly, it will remain unchanged.

Hence $f(x)\equiv\{x-(\alpha-i\beta)\}\,Q'$,

which shows that $\alpha-i\beta$ is also a root of the equation $f(x)=0$.

The number of complex roots of an equation with real coefficients is even.

1·441. *Every equation of odd degree has at least one real root.*

This follows from the fact that complex roots like $\alpha+i\beta$ occur in conjugate pairs.

1·442. *If* $f(x)=0$ *with real coefficients, has a multiple complex root,* $\alpha+i\beta$ *of order* p, *then* $\alpha-i\beta$ *is also a multiple root of order* p.

The proof is simple. We know that complex roots of the form $a+ib$, $a-ib$ of an equation with real coefficients occur in conjugate pairs. If the multiplicity of each kind is not the same, the coefficients of the equation formed by taking them as roots will be complex, which is contrary to the hypothesis.

It follows from the above, that if $\alpha_r+i\beta_r$ $(r=1, 2,...q)$ be complex roots of multiplicity p_r $(r=1, 2,...q)$, of the equation $f(x)=0$, then

$$[(x-\alpha_1)^2+\beta_1{}^2]^{p_1}[(x-\alpha_2)^2+\beta_2{}^2]^{p_2}...[(x-\alpha_q)^2+\beta_q{}^2]^{p_q}$$

is a factor of $f(x)$. Thus *the equation can be reduced by*

$$=(p_1+p_2+...+p_q) \text{ dimensions.}$$

1·443. It may be pointed out that a quadratic and a cubic cannot have a multiple complex root, and that a quartic having a multiple complex root, must have two distinct pairs of equal complex roots. It may also be noticed that if a quartic has two distinct pairs of equal roots, they are either both real, or both complex; for, one cannot be real and the other complex.

1·45. *In an equation with rational coefficients, if, for instance,* $a+\sqrt{b}$ *(*b *not being a perfect square) is a root, so is* $a-\sqrt{b}$.

1·451. *If an equation with rational coefficients has an irrational root* $a+\sqrt{b}$ *of multiplicity* p *(where* b *is not a perfect square), then* $a-\sqrt{b}$ *is also a multiple root of order* p.

The proofs of these theorems are left to the student.

1·452. An equation with rational coefficients may have a root of higher irrationality than the square root, *e.g.* $a+\sqrt[3]{b}$, where b is not a perfect cube. Roots of this type occur in *groups*, which we may call conjugate groups. If $a+\sqrt[3]{b}$ is a root, $a+\omega\sqrt[3]{b}$ and $a+\omega^2\sqrt[3]{b}$ will be two other roots, and these two latter roots are complex.

Again, an equation may have a root consisting of more than one irrational part, the other roots can then be put down at once, as in Exs. 8, 9, and 12 below. Again, a root may consist of an irrational part and a complex part, as in Ex. 10 below, in which $\sqrt{2}+\sqrt{-1}$ is one root. The other roots will be seen to be $\sqrt{2}-\sqrt{-1}$, $-\sqrt{2}+\sqrt{-1}$, $-\sqrt{2}-\sqrt{-1}$. Thus, we see that in *all such cases the roots of the equation occur in conjugate groups.*

1·5. Common roots of equations.

Let $f_n(x)=0$ and $f_m(x)=0$ be two equations of degrees n and m, respectively. If $h_1, h_2...h_k$, $(k<m$ or $n)$ be the roots of both the equations, then both $f_n(x)$ and $f_m(x)$ are divisible by $\phi(x)\equiv(x-h_1)(x-h_2)...(x-h_k)$, which is, therefore, the

H.C.F. (highest common factor) of $f_n(x)$ and $f_m(x)$. Thus, the common roots of $f_m(x)=0$, and $f_n(x)=0$ are the roots of the equation $\phi(x)=0$, where $\phi(x)$ is the H.C.F. of $f_n(x)$ and $f_m(x)$.

EXAMPLES I (*c*)

1. Find the cubic with real coefficients, which vanishes for $x=3+\sqrt{-5}$, and has the value -9, when x has the value 1.

2. Find the lowest degree equation with rational coefficients whose roots are $\sqrt{3}-1$ and $2-\sqrt{5}$.

3. Solve the equations :—

(*i*) $x^4-8x^3-12x^2+60x+63=0$, given one root equal to $3-\sqrt{30}$.

(*ii*) $x^4+x^3-25x^2+41x+66=0$, given one root equal to $3+i\sqrt{2}$.

4. Solve the equation $x^3+6x+20=0$, one root being $1+3i$.

5. Find the real factors of $x^3+4x^2+9x+10$, one of the factors being $x+1-2i$.

6. Show that the equation

$$x^6+2x^5-10x^4-12x^3+59x^2+10x-50=0$$

is satisfied by $x=i-3$ and $x=i+2$, and find the other roots.

(*P. U. 1922*)

7. Solve the equation :—

$$x^6-x^5-8x^4+2x^3+21x^2-9x-54=0,$$

given that one root is $\sqrt{2}+i$.

[**Hint :** $\sqrt{2}-i$, $-\sqrt{2}+i$, $-\sqrt{2}-i$ are the other three roots.]

8. Solve the equation

$$x^6-4x^5-11x^4+40x^3+11x^2-4x-1=0,$$

given that one root is $\sqrt{2}+\sqrt{3}$.

9. One root of $3x^5-4x^4-42x^3+56x^2+27x-36=0$ is $\sqrt{2}-\sqrt{5}$, find the remainining roots.

10. One root of the equation

$$2x^6-3x^5+5x^4+6x^3-27x+81=0$$

is $\sqrt{2}+\sqrt{-1}$, find the remaining roots.

11. Solve the equation $x^4-x^3-9x^2-14x+8=0$, given that one root is $-1+\sqrt[3]{3}$.

12. Form an equation with rational coefficients which shall have for a root the irrational expression $\sqrt{p}+\sqrt{q}$, where p and q are not perfect squares.

The expression has four different values according to the signs of the radicals, *i.e.*

$$\sqrt{p}+\sqrt{q},\ -\sqrt{p}+\sqrt{q},\ \sqrt{p}-\sqrt{q},\ \sqrt{p}-\sqrt{q}.$$

The required equation is, therefore,

$(x-\sqrt{p}-\sqrt{q})(x+\sqrt{p}+\sqrt{q})(x-\sqrt{p}+\sqrt{q})(x+\sqrt{p}-\sqrt{q})=0,$

i.e. $x^4-2(p+q)x^2+(p-q)^2=0.$

Aliter: Put $x=\sqrt{p}+\sqrt{q}$. Squaring both sides,

$$x^2=(p+q)+2\sqrt{pq}.$$

Hence $(x^2-p-q)^2=4pq$, which is the required equations.

13. Find the simplest equation with integral coefficients which has $-\sqrt{\frac{2}{3}}+\sqrt{\frac{3}{5}}$ and $-\sqrt{\frac{2}{3}}-\sqrt{\frac{3}{5}}$ among its roots. What are the other roots of the equation ? (*Peterhouse, etc., 1931*)

14. Show that the equation

$$\frac{A_1^2}{x-a_1}+\frac{A_2^2}{x-a_2}+\ldots\ldots+\frac{A_n^2}{x-a_n}=x+l,$$

where $A_1, A_2,\ldots A_n, a_1, a_2,\ldots\ldots a_n$ and l are real, cannot have imaginary roots.

[**Solution.** If possible, let $\alpha+i\beta$ be a root, then $\alpha-i\beta$ is also a root. Hence

$$(i)\ \frac{A_1^2}{(\alpha-a_1)-i\beta}+\frac{A_2^2}{(\alpha-a_2)-i\beta}+\ldots\ldots+\frac{A_n^2}{(\alpha-a_n)-i\beta}=(\alpha-i\beta)+l$$

$$(ii)\ \frac{A_1^2}{(\alpha-a_1)+i\beta}+\ldots\ldots+\frac{A_n^2}{(\alpha-a_n)+i\beta}=(\alpha+i\beta)+l.$$

Subtracting (ii) from (i),

$$2i\beta\left[\frac{A_1^2}{(\alpha-a_1)^2+\beta^2}+\ldots\ldots+\frac{A_n^2}{(\alpha-a_n)^2+\beta^2}+1\right]=0,$$

which is impossible, unless $\beta=0$.]

15. If the roots of the equation $x^n-1=0$ are 1, α, β, $\gamma\ldots$ show that $(1-\alpha)(1-\beta)(1-\gamma)\ldots\ldots=n$.

16. If $x_1, x_2,\ldots\ldots x_n$, are the roots of the equation

$$(a_1-x)(a_2-x)\ldots\ldots(a_n-x)+k=0,$$

then $a_1, a_2,\ldots\ldots a_n$, are the roots of the equation

$$(x_1-x)(x_2-x)\ldots\ldots(x_n-x)-k=0. \quad (\textit{Radford})$$

[Hint : $(a_1-x)(a_2-x)\ldots(a_n-x)+k\equiv(x_1-x)(x_2-x)\ldots(x_n-x)$.]

17. For how many values of x can a polynomial of degree n equal a constant k ?

18. Prove that $x=\frac{2}{n+1}$ satisfies the equation

$$(x-1)^3+(2x-1)^3+\ldots\ldots\ldots+(nx-1)^3=0,$$

and find the quadratic equation satisfied by the other two roots. (*Downing, 1928*)

19. Find the common roots of the equations

$x^4+3x^3-5x^2-6x-8=0$ and $x^4+x^3-9x^2+10x-8=0$.

20. Solve the equations

$$4x^4+12x^3-x^2-15x=0,\ 6x^4+13x^3-4x^2-15x=0\ ;$$

given that they have common roots.

Elementary Symmetric Functions

1·6. It is known that when α_1, α_2 are the roots of the equation $a_0x^2+a_1x+a_2=0$, the following relations hold :—

$$\alpha_1+\alpha_2=-\frac{a_1}{a_0},\quad \alpha_1\alpha_2=\frac{a_2}{a_0}.$$

Now we shall establish such relations for the general equation

$$f(x)\equiv a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots\ldots\ldots+a_{n-1}x+a_n=0\ldots\ldots\ldots(1)$$

It has been proved that if α_1, α_2, α_3,...α_n are the roots of this equation,

$$f(x)\equiv a_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots\ldots(x-\alpha_{n-1})(x-\alpha_n).$$

Multiplying out all the linear factors, we get the expression

$$a_0\{x^n-(\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n)x^{n-1}+(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3+\cdots)x^{n-2}$$
$$-(\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_4\ldots)x^{n-3}+\cdots+(-1)^n\alpha_1\alpha_2\alpha_3\ldots\alpha_n\}.$$

Now $\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n$, which is the sum of all the roots, may be represented shortly by $\Sigma\alpha_1$; in the same way $\alpha_1\alpha_2+\alpha_1\alpha_3+\cdots+\alpha_2\alpha_3+\cdots$, which is the sum of products of roots, taken two at a time, may be represented by $\Sigma\alpha_1\alpha_2$, and so on. Thus the expanded expression may be written as

$$a_0\{x^n-\Sigma\alpha_1x^{n-1}+\Sigma\alpha_1\alpha_2x^{n-2}-\Sigma\alpha_1\alpha_2\alpha_3x^{n-3}+\ldots+(-1)^n\alpha_1\alpha_2\alpha_3\ldots\alpha_n\},$$

the absolute term being the product of all the roots.

Now since the expression is identically equal to

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n,$$

the co-efficients of the corresponding powers of x must be equal.

Hence we get the following n relations :—

$$\left.\begin{aligned}\Sigma\alpha_1&=-\frac{a_1}{a_0},\\ \Sigma\alpha_1\alpha_2&=\frac{a_2}{a_0},\\ \Sigma\alpha_1\alpha_2\alpha_3&=-\frac{a_3}{a_0},\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ \alpha_1\alpha_2\alpha_3\ldots\alpha&=(-1)^n.\frac{a_n}{a_0}.\end{aligned}\right\}\quad(2)$$

It should be noted that in the above relations the coefficients of the alternate terms enter with their signs changed, the first change occurring at x^{n-1}.

1·61. If the coefficient of the highest term is unity, *i.e.* if $a_0=1$, the relations become simpler, there being no necessity of division by the coefficient of the first term. In that case,

$$\alpha_1\alpha_2.....\alpha_n=(-1)^n a_n,$$

which shows that the absolute term is exactly divisible by each root.

EXAMPLES I (*d*)

1. The sum of two roots of the equation

$$x^3-3x^2-16x+48=0,$$

is zero, find all the roots.

Let the roots be α, β, γ; then we have the following three relations between the roots and coefficients :—

$$\alpha+\beta+\gamma=3, \quad ..... \ (i)$$
$$\alpha\beta+\beta\gamma+\gamma\alpha=-16, \quad ..... \ (ii)$$
$$\alpha\beta\gamma=-48. \quad ..... \ (iii)$$

Let the additional relation here be written as

$$\alpha+\beta=0. \quad ..... \ (iv)$$

From (i) and (iv), $\gamma=3$. Substituting this value of γ in (iii),

$$\alpha\beta=-16 \quad .... \ (v)$$

This could have also been obtained from (ii) and (iv).

Now from (iv) and (v), α and β are found to be 4 and -4.

Thus the roots are 4, -4, 3.

The student will also see that after one root has been obtained to be 3, the cubic could have been divided by $x-3$, giving $x^2-16=0$, which gives $x=\pm 4$.

It is clear that *all* the relations between the roots and the coefficients are not necessary for solving the equation completely. The relation not used may be utilised for checking the results.

2. Solve the following equations, the sum of two of the roots being equal to zero, in each case :—

(i) $25x^3+50x^2-x-2=0$,

(ii) $4x^4-24x^3+31x^2+6x-8=0$,

(iii) $6x^4-3x^3+8x^2-x+2=0$. (*Math. Trip. I, 1932*)

3. The cubic $3x^3-11x^2+8x+k=0$, has two equal roots, find k and solve the equation.

Let the roots be α, α; β. Then we have

$$2\alpha+\beta=\frac{11}{3}, \quad ... \ (1)$$

$$\alpha^2+2\alpha\beta=\frac{8}{3}, \quad ... \ (2)$$

$$\alpha^2\beta=\frac{k}{3}. \quad ... \ (3)$$

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Substituting in (2) for β, $\frac{11}{3}-2a$, obtained from (1), we have the quadratic equation $9a^2-22a+8=0$, which gives $a=2$, or $\frac{4}{9}$. The corresponding values of β are $-\frac{1}{3}$ and $\frac{25}{9}$. Thus we get the two sets of values, $2, 2. -\frac{1}{3}; \frac{4}{9}, \frac{4}{9}, \frac{25}{9}$.

For the first set of roots, $k=4$; and for the second set $k=-\frac{400}{243}$.

4. The cubic $x^3-12x^2+45x-k=0$ has two equal roots. Find k and solve the equation.

5. Each of the cubics $2x^3-9x^2+12x-5=0$ and $2x^3-9x^2+12x-4=0$ has two equal roots. Find all the roots, and clear the ambiguity, if it arises anywhere in the solution.

6. Find the condition in order that the equation $x^3+3qx+r=0$ may have a pair of equal roots.

7. Show that the equation $x^3-3x+a=0$ has equal roots, if $a=\pm 2$. Solve the equation in each case.

8. Solve $x^4-8x^3+21x^2-20x+5=0$, it being given that the sum of two of the roots is equal to the sum of the other two.

(*P. U. 1918*)

The equation is identical with

$$(x^2-4x+p)(x^2-4x+q)=0,$$

i.e. with $x^4-8x^3+(p+q+16)x^2-4(p+q)x+pq=0$.

$$\therefore \quad p+q=5,\ pq=5.$$

Hence p, q are the roots of the equation $t^2-5t+5=0$, etc.

9. Find the condition that the roots of the equation $x^4+2ax^3+bx^2+cx+d=0$ be so related that the sum of two of the roots is equal to the sum of the other two.

Hence solve the equation $x^4-10x^3+27x^2-10x-16=0$.

Solution. If $\alpha, \beta, \gamma, \delta$, be the roots, $\alpha+\beta=\gamma+\delta=-a$.

Hence α, β ; γ, δ, are the roots of the equations

$$x^2+ax+p=0, \quad x^2+ax+q=0.$$

Hence $x^4+2ax^3+bx^2+cx+d\equiv(x^2+ax+p)(x^2+ax+q)$.

Comparing coefficients, $a^2+(p+q)=b$, $a(p+q)=c$.

Elimination of $p+q$ gives the required condition $a^3-ab+c=0$.

10. Solve the equation $x^4+2x^3-21x^2-22x+40=0$, given that the sum of two roots is equal to the sum of the other two.

11. Solve the above equation of Ex. 8, given that the product of two of the roots is 5.

Since $\gamma\delta=5$, $\therefore\ \alpha\beta=1$: hence the biquadratic can be factorized in the form $(x^2+px+1)(x^2+qx+5)$.

$\therefore\ x^4-8x^3+21x^2-20x+5\equiv(x^2+px+1)(x^2+qx+5)$.

Comparing coefficients,

$$-8=p+q, \quad \ldots\ (i)$$
$$21=6+pq, \quad \ldots\ (ii)$$
$$-20=5p+q. \quad \ldots\ (iii)$$

From (i) and (iii), $p=-3$, $q=-5$. Hence the roots of the biquadratic are the roots of the two quadratics

$$x^2-3x+1=0 \text{ and } x^2-5x+5=0,$$

which give $\dfrac{3\pm\sqrt{5}}{2}$ and $\dfrac{5\pm\sqrt{5}}{2}$.

12. Given that the product of two of the roots of the equation $x^4-10x^3+42x^2-82x+65=0$ is 13, solve the equation completely.

13. If the product of two of the roots of $x^4+px^3+qx^2+rx+s=0$ is equal to the product of the other two, show that $r^2=p^2s$.

Show that this condition is satisfied if the roots are in G. P.

14. Show that if $c^2=a^2d$, then the product of two of the roots of the equation $x^4+ax^3+bx^2+cx+d=0$ is equal to the product of the other two.

Hence solve the equation $x^4+x^3+2x^2+2x+4=0$.

15. The equation $x^4-2x^3+4x^2+6x-21=0$ has two roots equal in magnitude and opposite in sign : find all the roots.

(Queen's College)

16. Given that sum of two of the roots is zero, solve completely the equation $4x^4+8x^3+13x^2+2x+3=0$.

(St. Catherine, 1937)

17. Given that two of the roots of

$$45x^4-54x^3-98x^2+150x-75=0,$$

are equal in absolute value but opposed in sign, complete the solution of the equation. *(Math. Trip. I, 1926)*

18. Solve the equation $6x^4-3x^3+8x^2-x+2=0$, being given that it has a pair of roots whose sum its zero. *(Math. Trip. 1932)*

19. Find the conditions that the general biquadratic equation $ax^4+4bx^3+6cx^2+4d+e=0$ may have its roots (i) equal in pairs, (ii) all equal. *(P. U. 1917, 1922 and 1925)*

For (i), we take the roots α, α, β, β, and get

$$\alpha+\beta=-\frac{2b}{a},\quad \alpha^2+\beta^2+4\,\alpha\beta=\frac{6c}{a},$$

$$\alpha\beta(\alpha+\beta)=-\frac{2d}{a}, \text{ and } \alpha^2\beta^2=\frac{c}{a}.$$

Substituting the value of $\alpha+\beta$ from the first relation in the third, we have $\alpha\beta=\frac{d}{b}$, which value substituted in the fourth relation, gives $ad^2-eb^2=0$. ... ... (A)

Again, from the second relation similarly is obtained the condition $2b^3+a^2d=3abc$ ... ... (B)

Thus, in order that the roots may be equal in pairs, conditions (A) & (B) should hold.

Here the relations between the coefficients and the roots to be consistent, require that two conditions should be fulfilled.

20. Solve the equation

$$x^4+4x^3-2x^2-12x+9=0$$

given that it has two pairs of equal roots.

Let the roots be α, α, β, β,

$\therefore \quad 2(\alpha+\beta)=-4 \qquad \alpha^2\beta^2=9$

$\therefore \quad \alpha+\beta=-2$ and $\alpha\beta=\pm 3$.

Thus α and β are the roots of the equation $x^2+2x\pm 3=0$.

Hence put tentatively $(x^2+2x\pm 3)^2=x^4+4x^3-2x^2-12x+9$.

Comparing the coefficients on both sides, it is seen that the positive sign before 3 is to be rejected. Thus α and β are given by the equation $x^2+2x-3=0$

Hence the roots are 1, 1, -3, -3.

21. The biquadratic $x^4-14x^3+73x^2-168x+144=0$ has two pairs of equal roots ; find them.

22. The roots of the cubic equation $4x^3+12x^2+11x+3=0$ are in Arithmetic Progression, find them.

[Take the roots as $a-k$, a and $a+k$.]

23. Solve the equation $x^3-12x^2+39x-28=0$, whose roots are in Arithmetical Progression. (*P. U. 1940*)

24. The roots of the equation

$$16x^4-64x^3+56x^2+16x-15=0$$

are known to be in Arithmetical Progression ; solve the equation (*Math., Trip. I, 1931*)

25. The roots of the equation $8x^3-14x^2+7x-1=0$ are in Geometrical Progression ; find them.

[Take the roots as $\frac{a}{r}$, a, ar.]

26. Solve the equation $27x^3+42x^2-28x-8=0$ whose roots are in Geometric Progression. (*P. U. 1942*)

27. The roots of the equation $2x^4-45x^3+315x^2-810x+648=0$ are in Geometrical Progression : find them.

[The product of two of the roots is equal to the product of the other two, now proceed as in Ex. 14.]

28. Solve the equation $x^4+20x^3-210x^2-540x+729=0$, the roots being in Geometrical Progression.

29. Two roots of the equation $x^4+6x^3-16x^2+24x-80=0$ are pure imaginary. Find them.

30. Solve the equation $x^3-9x^2+14x+24=0$, two of whose roots are in the ratio 3 : 2.

31. One root of $x^3-px^2+qx-r=0$ is double the other, show that it may be found from a quadratic equation.

32. One root of the cubic $x^3-px^2+qx-r=0$ is n times the other, show that it may be found from a quadratic equation.

[Let the roots be α, $n\alpha$, β, then

$$\alpha(n+1)+\beta=p,\ n\alpha^2+\beta\alpha(n+1)=q.$$

Substituting for β in the second, we see that α is one of the roots of the quadratic

$$(n^2+n+1)x^2-p(n+1)x+q=0.$$

The value of α should be so chosen that the value of β obtained from one of the equations

$$\alpha(n+1)+\beta=p,\ n\alpha^2\beta=r,$$

satisfies the other.]

33. Show that the sum of two of the roots of the equation $x^4+px^3+qx^2+r=0$ will be equal to zero, if $r^2-pqr+sp^2=0$.

34. Show that the four roots of the equation

$$x^4+px^3+qx^2+rx+s=0,$$

will be connected by the relation, $\alpha\beta+\gamma\delta=0$, if $p^2s+r^2=4qs$.

35. Prove that the sum of the n nth roots of unity is always zero.

36. Prove that the equation $ax^5+bx^3+cx^2+d=0$ will have three roots equal, if $-\frac{c}{b}=\frac{b^2}{5ac}=\frac{5bd}{c^2}$, each of these quantities being equal to the repeated root.

Solve $32x^5-360x^3+540x^2-243=0$. (*P. U. 1923*)

37. The product of two of the roots of the cubic equation $am^3+(2a-x)\,m+y=0$, in m, is a constant k; find the relation between the coefficients that must hold.

[Let m_1, m_2, m_3, be the roots, and let $m_1 m_2 = k$.
We have

$$m_1+m_2+m_3=0, \qquad \ldots\ldots(i)$$

$$m_1m_2+m_2m_3+m_3m_1=\frac{2a-x}{a}, \qquad \ldots\ldots(ii)$$

$$m_1m_2m_3=-\frac{y}{a}, \qquad \ldots\ldots(iii)$$

$$m_1m_2=k, \qquad \ldots(iv), \text{ the given relation.}$$

From (iii) and (iv), $m_3=-\frac{y}{ka}$.

Here it is not necessary to eliminate m_1, m_2, m_3 with the help of relations (i) and (ii). The easiest method is to substitute the value of m_3 in the given equation, which must be satisfied by it.

Thus we have, $y\{y^2+k^2a(2a-x)-k^3a^2\}=0$, $\ldots\ldots(v)$
which is the required relation.

This gives the solution of an important problem in Co-ordinate Geometry which may be stated thus :—

The locus of the point of intersection of two normals to the parabola $y^2-4ax=0$, the product of whose slopes is a constant k, is another parabola $y^2+k^2a(2a-x)-k^3a^2=0$.]

38. The roots of the cubic $x^3+3px+3qx+r=0$ are in (*i*) A. P., (*ii*) G. P., (*iii*) H. P., if (*i*) $2p^3-3pq+r=0$, (*ii*) $rp^3=q^3$, (*iii*) $2q^3=r(3pq-r)$.
Solve the equation in the three cases.

39. One root of the equation $a_0x^3+a_1x^2+a_2x+a_3=0$ is equal to the sum of other two. Solve it and find the relation among the co-efficients.
Solve the equation $x^3+2x^2+3x+2=0$.

40. The ratio of two of the roots of the biquadratic

$$x^4-12x^3+47x^2-72x+36=0$$

is equal to the ratio of the other two. Solve the equation.

41. Show that if the roots of the equation

$$x^n+np_1x^{n-1}+\frac{n(n-1)}{2!}p_2x^{n-2}+\cdots\cdots=0$$

be in A. P., they can be obtained from

$$-p_1-\left\{\frac{3(p_1^2-p_2)}{n+1}\right\}^{\frac{1}{2}}(n-2k+1), \text{ for } k=1, 2, \ldots\ldots n.$$

[Sol. Suppose that the roots are

$$a_1=a,\ a_2=a+d\ldots\ldots a_n=a+\overline{n-1}\,d.$$

$$\therefore\ -np_1=\Sigma a_1=\frac{n}{2}\{2a+\overline{n-1}d\}$$

$$\therefore\ -p_1=a+\tfrac{1}{2}\overline{n-1}\ d \qquad \ldots\ldots(i)$$

$$\Sigma a_1^{\,2}=(\Sigma a_1)^2-2\ \Sigma a_1\ a_2=n^2p_1^{\,2}-n(n-1)p_2\,;$$

$$\therefore\ n^2p_1^{\,2}-n(n-1)p_2=a^2+(a+d)^2+\ldots\ldots+(a+\overline{n-1}\ d)^2$$

$$=na^2+2ad(1+2+\ldots\ldots+\overline{n-1})$$

$$+(1^2+2^2+\ldots\ldots+\overline{n-1}^{\,2})d$$

$$=na^2+2ad\,\frac{(n-1)n}{2}+\frac{n(n-1)(2n-1)}{6}d^2,$$

or $$np_1^{\,2}-(n-1)p_2=a^2+ad(n-1)+\frac{(n-1)(2n-1)}{6}d^2 \qquad \ldots\ldots(ii)$$

From (ii) subtract the square of (i);

then $$(n-1)(p_1^{\,2}-p_2)=\frac{(n^2-1)d^2}{12}\ ;$$

$$\therefore\ d=2\left\{\frac{3(p_1^{\,2}-p_2)}{n+1}\right\}^{\frac{1}{2}}.$$

Now $$a_k=a+\overline{k-1}\ d$$

$$=a+\frac{n-1}{2}d+\tfrac{1}{2}(2k-1-n)d$$

$$=-p_1-\ (n+1-2k)\left[\frac{3(p_1^{\,2}-p_2)}{n+1}\right]^{\frac{1}{2}}.$$

42. If $z^4+6\mathrm{H}z^2+4\mathrm{G}z+a_0^{\,2}\mathrm{I}-3\mathrm{H}^2=0$ has two pairs of equal roots, prove that

$$\mathrm{G}=0,\ a_0^{\,2}\mathrm{I}=12\mathrm{H} \qquad (P.\ U.\ 1926)$$

43. If $x+ay+a^2z=a^3$, $x+by+b^2z=b^3$, $x+cy+c^2z=c^3$, find x, y, z.

[a, b, c are the roots of $t^3=t^2z+ty+x$. Hence etc.]

CHAPTER II

DETERMINANTS

2·1. Definitions.

Permutations. Let 1, 2, 3, 4,...n, (or a, b, c,...) be integers (or letters), which stand in natural order. Any other arrangement (*e.g.*, 4, 7, 6, 3, 2,... or *edbac*...) of these numbers (or letters) is called a **permutation.** A number (or letter) q is called higher than p if it occurs to the right of p in the natural order.

Transposition. An interchange of two numbers (or letters) is called a **transposition.**

Inversion. If in a permutation a higher number precedes a lower one, an **inversion** is said to occur.

2·11. Class of a permutation. A permutation is said to be **even**, or of **even class**, if it contains an even number of inversions. If the number of inversions be odd, it is said to be **odd**, or of **odd class.**

The class of a permutation can be determined by actually counting the inversions. Form the pairs of each number with those which follow it and count the inversions in all pairs. Their sum gives the total number of inversions in the permutation.

Take, for instance, the permutation 364152. We form the following couplets (the inversions will be indicated by crooked brackets).

The couplets (36), (34), {31}, (35), {32} give 2 inversions.

The couplets {64}, {61}, {65}, {62} give 4 inversions.

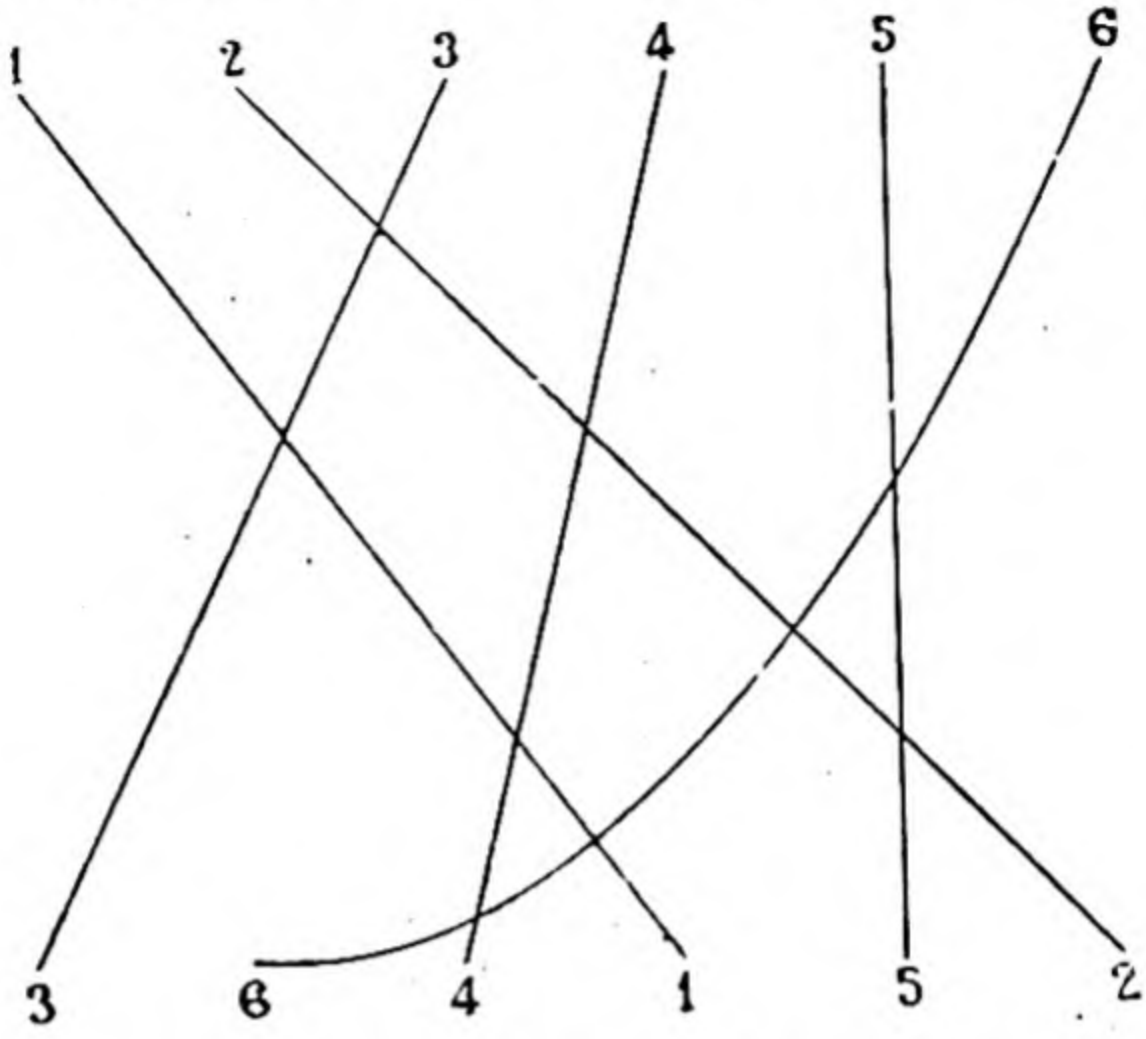
The couplets {41}, (45), {42} contain 2 inversions.

The couplets (15), (12) have no inversion,

The couplet {52} is an inversion.

The total number of inversions is 9. The class of the permutation is odd.

The same thing can be exhibited graphically as follows :—



In the upper line write the six numbers in the natural order, and below it, in the second line, write the permutation. Join each number of the upper line with the same number in the lower line by lines (straight or curved), so that no three lines pass through the same point, and no two lines intersect in more than one point. The number of intersections which lie between the two rows, equals the number of inversions.

2·12. The effect of a transposition. *A single transposition of two elements in a permutation introduces an odd number of inversions, and therefore changes the class of the permutation.*

Take, for instance, the permutation 3 1 9 8 2 6 5 7 4,
and, suppose, we interchange 9 and 6, and get the permutation
3 1 6 8 2 9 5 7 4.

We may note that the interchange of 9 and 6 has no effect on the inversions of the groups, 31 which precedes 9, and 574 which follows 6. We may, therefore, consider the effect only on the group 9826. We may also note that by an interchange of two consecutive numbers we either gain or lose an inversion (*e.g.*, by the interchange of 9, 8 which gives 89 an inversion is lost, and the interchange of 2 and 6 gains an inversion). Now 9 can be brought to the place of 6 by passing over 8,2,6, *i.e.* three inversions are lost (in general, m, say), and then 6 can be brought to the original position of 9 by two interchanges (in general, $m-1$), so that 2 inversions are gained or lost (in general, $m-1$). Thus the number of inversions gained, or lost, is $2m-1$.

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2·13. Consider the two simultaneous equations :—

$$\left.\begin{aligned} a_1x+b_1y=0 \\ a_2x+b_2y=0 \end{aligned}\right\} \quad \ldots (2)$$

and

Elimination of x and y gives $a_1b_2-a_2b_1=0$.

The expression $a_1b_2-a_2b_1$, ... (2)

which is a function of the co-efficients a_1, b_1, a_2, b_2, is symbolically represented by

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \quad \ldots (3)$$

which is called a **Determinant** of the second order, and $a_1b_2-a_2b_1$ is known as the **development** or **expansion** of the determinant, while a_1, a_2, b_1, b_2 are called the **elements** of the determinant.

Next, we take the three simultaneous equations :—

$$\left.\begin{aligned} a_1x+b_1y+c_1z=0, \\ a_2x+b_2y+c_2z=0, \\ a_3x+b_3y+c_3z=0. \end{aligned}\right\} \quad \ldots (1)$$

and

From the second and third of these equations, the proportional values of x, y, and z are easily found to be

$$\frac{x}{b_2c_3-b_3c_2}=\frac{y}{c_2a_3-a_2c_3}=\frac{z}{a_2b_3-a_3b_2}.$$

Now x, y, and z may be eliminated by putting the proportional values in the first of equations. ... (1)

Thus we get the eliminant

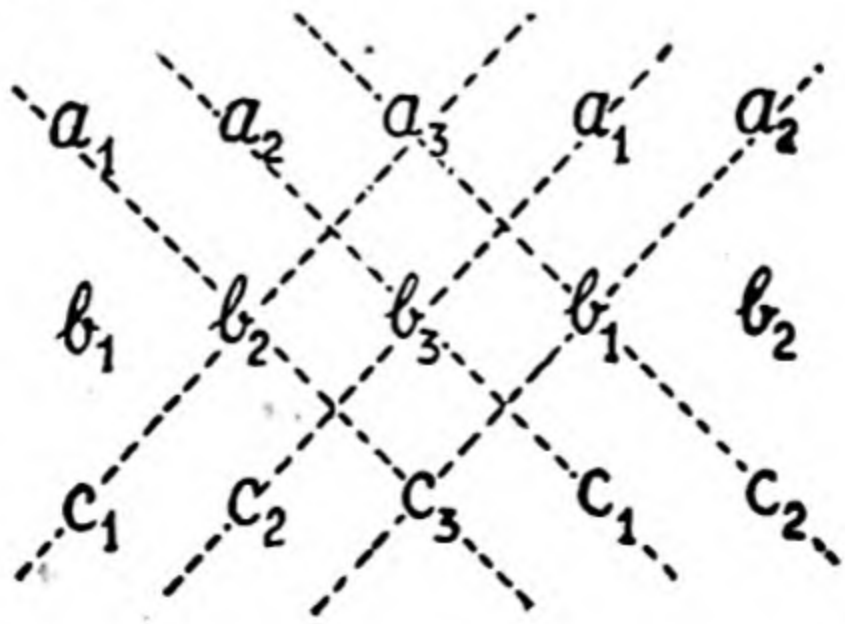
$$a_1(b_2c_3-b_3c_2)-b_1(a_2c_3-a_3c_2)+c_1(a_2b_3-a_3b_2)=0. \quad \ldots (2)$$

The expression on the left-hand side of (2), which is a function of the nine **elements**, or **constituents**, a_1, b_1, c_1, $a_2 \ldots c_3$, is the *expansion* of the *determinant of the third order*,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \quad \ldots (3)$$

The following rule of expansion is due to Sarrus.

Write down three columns of the determinant $(a_1b_2c_3)$, and re-write thc 1st two.



The positive terms in the expansion are obtained from the numbers on the lines, running diagonally from top to bottom, and the negative terms from the numbers on the lines running diagonally from bottom to top, beginning being made, in each case, from an element in the first column.

EXAMPLES II (*a*)

1. $$\begin{vmatrix} 1 & a & b \\ -a & 1 & c \\ -b & -c & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & c \\ -c & 1 \end{vmatrix} + a\begin{vmatrix} a & b \\ -c & \end{vmatrix} - \begin{vmatrix} a & b \\ 1 & c \end{vmatrix}.$$

[expanding by means of the first column]

$$=(1+c^2)+a(a+bc)-b(ac-b)=1+a^2+b^2+c^2.$$

2. Show that the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc-af^2-bg^2-ch^2+2fgh.$$

3. Expand and simplify the following :—

$$(i)\ \begin{vmatrix} 1 & x & y \\ 0 & \cos x & \sin y \\ 0 & \sin x & \cos y \end{vmatrix}, \quad (ii)\ \begin{vmatrix} y-x & -x^2+xy-y^2 \\ x+y & x^2+xy+y^2 \end{vmatrix}.$$

[It is obviously convenient to expand (*i*) from the first column.]

4. Find the values of the determinants :

$$(i)\ \begin{vmatrix} 1 & 5 & 9 \\ 3 & 8 & 4 \\ 6 & 2 & 7 \end{vmatrix}, \quad (ii)\ \begin{vmatrix} 3 & 0 & -5 \\ 7 & -1 & 0 \\ -9 & 6 & 1 \end{vmatrix}, \quad (iii)\ \begin{vmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{vmatrix}.$$

The last determinant is an example of a 'magic square'. The sum of every row, column, and diagonal line is the same.

5. Write down, in determinant form, the result of eliminating x, y, z, from the equations :—

$$lx+my+nz=0,\ nx+ly+mz=0,\ mx+ny+lz=0.$$

Hence show that the eliminant is $l^3+m^3+n^3-3lmn=0$.

2·14. The examples of Art. 2·13 exhibit the following properties which will enable us to formulate the definition of a determinant of any order n.

1. A determinant of order n is an integral function of n^2 elements (also called constituents), arranged in the form of a solid square, along n horizontal lines (called rows) and n vertical lines (called columns).

2. Each term in the expansion is the product of n factors, in which each of the letters, a, b, etc., and each of the suffixes 1, 2, ...n, occurs once, and once only. If in each term the letters a, b, etc., be arranged in order, the suffixes will be some permutation of the natural order 1, 2, 3, etc., of the suffixes.

3. The number of terms is n !, which is the number of permutations of 1, 2,...n. Half of the terms are positive, and the remaining half are negative.

4. A term is positive, or negative, according as the permutation of suffixes is even, or odd.

Ex. Take a determinent of the fourth order and obtain its expansion by the above rules.

2·15. Definition. The above discussion enables us to formulate the following definition of a determinant :—

The sum of n ! different terms, viz.,

$$\Sigma \pm a_1 b_2 c_3 \ldots s_n,$$

obtained by permuting the suffixes 1, 2,...n, *in all possible ways, the order of the letters* a, b,...s *being kept intact, and the sign of a term being positive or negative according as the permutation of suffixes is even or odd, constitutes a determinant of order n. The sign of the term* $a_1 b_2 c_3 \ldots s_n$ *is positive.* This determinant is symbolically represented by

$$\begin{vmatrix} a_1 & b_1 & c_1 \ldots\ldots s_1 \\ a_2 & b_2 & c_2 \ldots\ldots s_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_n & b_n & c_n \ldots\ldots s_n \end{vmatrix},$$

and will here be denoted by Δ.

It is briefly written as $| a_1 b_2 c_3 \ldots s_n |$, or as $(a_1 b_2 c_3 \ldots s_n)$, or simply as $(abc \ldots s)$.

The term $a_1 b_2 c_3 \ldots s_n$, which is obtained by multiplying the elements situated in the diagonal drawn from the left-hand top-corner, is called the **leading** or **principal term.**

A determinant of order n is, therefore, an integral function of n^2 elements, arranged in the form of a square along n horizontal lines and n vertical lines. The horizontal lines are called *rows* and the vertical lines *columns*. The letters signify the columns and the suffixes, 1, 2,...refer to 1st. 2nd...rows. Thus d_i is the element occurring in the fourth column and the i^{th} row.

Another notation for the determinant of the nth order is the following :—

$$\begin{vmatrix} a_{11} & a_{12} & a_{13}.....a_{1n} \\ a_{21} & a_{22} & a_{23}.....a_{2n} \\ a_{31} & a_{32} & a_{33}.....a_{3n} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1} & a_{n2} & a_{n3}.....a_{nn} \end{vmatrix},$$

or $(a_{11}\, a_{22}\, a_{33}...a_{nn})$, or $\mid a \mid n, n$.

First suffixes refer to rows while the second refer to columns. Thus the element a_{ij} occurs in the ith row and the jth column.

Note. (*i*) *An interchange of two letters, keeping the suffixes fixed, is equivalent to an interchange of the suffixes, letters being kept unaltered.*

(*ii*) *Each term in the expansion contains one and only one element from each row and column.*

(*iii*) *The interchange of two complete elements, for instance, a_i, and b_j, in a term does not change the sign of the term. For, the interchange of a_i with b_j is equivalent to two interchanges; a with b, and i with j.*

As the suffixes refer to rows, $a_r, b_r,...s_r$ occur in the rth row. So, in the definition, the parts of the elements that are to be permuted are those which refer to rows.

Ex. Write down the symbolic notation for the determinant $(a_1b_2c_3d_4)$, and obtain all the terms from the notation according to the above rules, by keeping the order of the suffixes fixed, and permuting the letters.

2·21 Theorem I. *If rows be changed into columns and columns into rows, the determinant rémains unaltered.*

Throughout these theorems we shall call the given determinant Δ, and the new determinant Δ'.

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For example, for a determinant of the *second* order,

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1, \text{ which is obviously equal to } \Delta',$$

i.e.,
$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix},$$

Now in the general case, we have to show that

$$\begin{vmatrix} a_1 & b_1 & c_1 \ldots\ldots\ldots l_1 \\ a_2 & b_2 & c_2 \ldots\ldots\ldots l_2 \\ a_3 & b_3 & c_3 \ldots\ldots\ldots l_3 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_n & b_n & c_n \ldots\ldots\ldots l_n \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \ldots\ldots a_n \\ b_1 & b_2 & b_3 \ldots\ldots b_n \\ c_1 & c_2 & c_3 \ldots\ldots c_n \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ l_1 & l_2 & l_3 \ldots\ldots l_n \end{vmatrix}.$$

The expansions are obtained by taking one and only one element from each column, and one and only one from each row. Any element situated in the rth row and the sth column in Δ becomes simply an element in the sth row and the rth column with reference to Δ'; the terms, in magnitude, are, therefore, the same in Δ' as in Δ. The terms are also of the same sign. For, let $\pm a_\alpha b_\beta \ldots l_\rho$, where a, b,...l occur in order, while $(\alpha,\beta,\ldots\rho)$ is some permutation of (1, 2, 3,...n), be a term of Δ. The elements in the term can be re-arranged so that the suffixes are in order, and a,b,c...are a permutation; and this is a term of Δ'. The second is obtained from the principal term by the same number of interchanges of letters as the first is obtained by the interchanges of suffixes.

Thus every term in Δ' is a term of Δ, and has the same sign. The two determinants are, therefore, identical.

When a column or a row is spoken of in a general way, it is called a *line*. Since columns are situated parallel to each other, and similar is the case with rows, "*parallel lines*" is the common term applied to columns as well as to rows.

2·22. Theorem II. *If two parallel lines of a determinant are interchanged, the determinant retains its absolute value, but changes in sign.*

Proof. Evidently the terms in the expansions of both Δ and Δ' are the same in magnitude and number, for each term contains one and only one element from each row, and one and only one from each column. Further, an interchange of two lines is equivalent to an interchange of two suffixes, or two letters. Thus every term of Δ occurs in Δ', with the sign changed; hence $\Delta' = -\Delta$.

2·221. Cor. 1. If any line of Δ be passed over two parallel, lines, *i.e.*, if the resulting determinant is like

$$\begin{vmatrix} a_1 & c_1 & d_1 & b_1 \\ a_2 & c_2 & d_2 & b_2 \\ a_3 & c_3 & d_3 & b_3 \\ a_4 & c_4 & d_4 & b_4 \end{vmatrix} \quad \text{(when } \Delta \text{ is of fourth order),}$$

then $\Delta' = (-1)^2 \Delta = \Delta$.

Generally, *if any line of a determinant be passed over m parallel lines, the resulting determinant* $\Delta' = (-1)^m \Delta$.

It is also obvious that $\Delta = (-1)^m . \Delta'$.

2·222. *The sign of any term will be positive, or negative, according as that term can be made the principal term in Δ' by an even, or odd, number of movements of lines of Δ.*

Thus if we require the sign of $a_2 b_3 c_4 d_1$ in a determinant of the fourth order, Δ' will be

$$\begin{vmatrix} a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \\ a_1 & b_1 & c_1 & d_1 \end{vmatrix},$$

and this requires three movements of rows of Δ, *i.e.*, the first row is passed over the next three rows. Hence $a_2 b_3 c_4 d_1$ has a negative sign before it.

Ex. Obtain the result by considering the permutations of $a_1 b_2 c_3 d_4$ (*i*) by permuting the suffixes, and (*ii*) by permuting the letters, the suffixes keeping their order intact.

2·3. Minors and Co-factors.

Let $\Delta = | a_1 b_2 c_3 \ldots s_n |$.

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Definition 1. *The determinant that is left by removing the row and the column which intersect at* a_1, *is called the co-factor of* a_1, *and is denoted by* A_1.

To determine the co-factor of p_r which occurs in the rth row and pth column, we bring p_r to the left-hand top corner position by $r-1$ movements of the rth row upwards, and $p-1$ movements of the pth column to the left. Calling the transformed determinant Δ', we have

$$\Delta = \begin{vmatrix} p_r & a_r & b_r \ldots\ldots s_r \\ p_1 & a_1 & b_1 \ldots\ldots s_1 \\ p_2 & a_2 & b_2 \ldots\ldots s_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ p_n & a_n & b_n \ldots\ldots s_n \end{vmatrix} .(-1)^{p+r-2} = \Delta'(-1)^{p+r}.$$

Hence by the definition above, the *co-factor* P_r *of* p_r in Δ *is the co-factor of* p_r *in* Δ', *multiplied by* $(-1)^{p+r}$; hence it is

$$(-1)^{p+r} \begin{vmatrix} a_1 & b_1 & \ldots\ldots s_1 \\ a_2 & b_2 & \ldots\ldots s_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_n & b_n & \ldots\ldots s_n \end{vmatrix},$$ which is of order $n-1$.

Definition 2. *The co-factor* P_r *of* p_r *without the proper sign* $(-1)^{p+r}$, *is called the minor of* p_r, *and is denoted by* P'_r. *Thus the minor of* p_r *is the determinant obtained from* Δ *by removing the row and column that intersect at* p_r.

2·31. *Expansion of a determinant.*

$$\Delta = | a_1 b_2 \ldots\ldots s_n | = \Sigma \pm a_1 b_2 \ldots\ldots s_n.$$

Illustration. Consider a determinant of the third order, viz.

$$(a_1 b_2 c_3) \equiv \Sigma \pm a_1 b_2 c_3.$$

In this summation, group all the terms which contain (*i*) a_1, (*ii*) a_2, (*iii*) a_3. We have thus

$$(a_1 b_2 c_3) = a_1(b_2 c_3 - b_3 c_2) + a_2(b_3 c_1 - b_1 c_3) + a_3(b_1 c_2 - b_2 c_1).$$

We note that the coefficient of a_1, viz. $b_2c_3-b_3c_2$ does not contain the letter a, nor the suffix 1, and consists of the permutations of suffixes in b_2c_3, the letters being kept in order.

The coefficient of a_2 does not, similarly, contain the letter a, nor the suffix 2; also, 2 can be made the suffix of a by interchanging 1 and 2 in the leading term $a_1b_2c_3$, which thus gives $-a_2b_1c_3$. Thus the coefficient of $-a_2$, which is obtained by permuting 1 and 3 in b_1c_3, the order of the letters being kept fixed, is $b_1c_3-b_3c_1$. This gives the coefficient of a_2 as $b_3c_1-b_1c_3$, which is precisely A_2. In the same way, the coefficient of a_3 is A_3.

Now we take up the general case of the expansion of the determinant.

Proof. In the summation $\Sigma\pm a_1b_2\ldots s_n$, group all the terms which contain (*i*) a_1, (*ii*) $a_2\ldots(n)a_n$. Calling the coefficients of these T_1, T_2....., we have

$$|\, a_1b_2\ldots\ldots s_n \,| = a_1T_1+a_2T_2+\ldots\ldots+a_nT_n.$$

To determine T_1, we notice that it contains neither the letter a, nor the suffix 1. Thus T_1 consists of all permutations of $b_2\ c_3\ldots s_n$, where letters are kept fixed. Thus $T_1=\Sigma b_2c_3\ldots s_n$, and this is precisely A_1.

Now T_2 does not contain the letter a and the suffix 2. So 2 may be made the suffix of a by interchange of 1 and 2 in the leading term $a_1\ b_2\ldots s_n$; the result obtained is $-a_2b_1c_3\ldots s_n$. Thus the coefficient of $-a_2$ will be obtained by permuting the suffixes 1, 3,..., keeping the order of the letters intact. The required coefficient is $\Sigma\pm b_1\ c_3\ldots c_n$, which is precisely A'_2. Thus the coefficient of a_2 is $-A'_2=A_2$.

Again, to determine T_r which contains neither a nor r, the suffix r can be made the suffix of a by $r-1$ transpositions in the leading term. The result obtained is $(-1)^{r-1}\ a_rb_1c_2\ldots s_n$. The coefficient of $(-1)^{r-1}\ a_r$ is obtained by permuting 1, 2,...$(r-1)$, $(r+1)$,...n, the result being $\Sigma\pm b_1c_2\ldots s_n$, *i.e.*, A'_r. So the coefficient of a_r is $(-1)^{r-1}\ A'_r=A_r$.

Hence

$$\Delta=a_1A'_1-a_2A'+\ldots+(-1)^{r-1}a_r\ A'_r+\ldots+(-1)^{n-1}a_nA'_n,$$
$$\text{or}\quad a_1\ A_1+a_2A_2+\ldots+a_rA_r+\ldots+a_n\ A_n. \qquad \ldots(1)$$

Since any row or any column can be brought to the position of the first column, we have

$$\Delta=p_1\ P_1+p_2\ P_2+\ldots\ldots\ldots+p_nP_n$$
$$=a_rA_r+b_r\ B_r+\ldots\ldots\ldots+s_r\ S_r. \qquad \ldots(2)$$

EXAMPLES II (*b*)

1. Count the inversions in the following :—

(*i*) 263154, (*ii*) 7326451. (*iii*) 28631754 (*iv*) 784132965.

2. Determine the sign of the term $a_3\, b_4\, c_1\, d_2$ in a determinant of fourth order.

3. Evaluate the determinant

$$\Delta = \begin{vmatrix} 0 & 1 & 2 & 1 \\ 3 & 1 & 0 & 1 \\ 1 & 0 & 5 & 2 \\ 2 & 3 & 0 & 4 \end{vmatrix}.$$

Now $\Delta = -\begin{vmatrix} 3 & 0 & 1 \\ 1 & 5 & 2 \\ 2 & 0 & 4 \end{vmatrix} + 2\begin{vmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 2 & 3 & 4 \end{vmatrix} - \begin{vmatrix} 3 & 1 & 0 \\ 1 & 0 & 5 \\ 2 & 3 & 0 \end{vmatrix},$

(expanding in terms of first row). Expanding further these three determinants of the third order, the result is found to be -45.

Evaluate

4. $\begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix}$.

5. $\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$.

Find the values of the determinants

6. $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 0 \end{vmatrix}$.

7. $\begin{vmatrix} 5 & -2 & 1 & 0 \\ 3 & 1 & -2 & 4 \\ 2 & 0 & -1 & 2 \\ 1 & -1 & 2 & 3 \end{vmatrix}$.

8. Show that the determinant

$$\begin{vmatrix} a & a & a & 0 \\ b & b & 0 & b \\ c & 0 & c & c \\ 0 & d & d & d \end{vmatrix} = -3abcd.$$

Expand the following determinants :—

9. $\begin{vmatrix} 0 & c & b & l \\ c & 0 & a & m \\ b & a & 0 & n \\ l & m & n & 0 \end{vmatrix}$.

10. $\begin{vmatrix} a & \alpha & \beta & \gamma \\ -\alpha & a & l & -m \\ -\beta & -l & a & n \\ -\gamma & m & -n & a \end{vmatrix}$.

11. $\begin{vmatrix} -p & q & r & s \\ q & -p & s & r \\ r & s & -p & q \\ s & r & q & -p \end{vmatrix}$.

12. $\begin{vmatrix} 0 & c & b & x \\ -c & 0 & a & y \\ -b & -a & 0 & z \\ -x & -y & -z & 0 \end{vmatrix}$.

Show that the result in Ex. 12 may be put as

$$(ax-by+cz)^2.$$

2·32. Theorem III. *A determinant vanishes if two of its parallel lines are identical.*

It is clear that the interchange of identical lines leaves the determinant unchanged, yet by Theorem II, the interchange of two lines changes the sign of the determinant ; hence

$$\Delta = \Delta' = -\Delta, \text{ or } \Delta = 0.$$

Cor. It has been proved that

$$\Delta = a_r\ \mathrm{A}_r + b_r\ \mathrm{B}_r + \ldots\ldots + s_r\ \mathrm{S}_r$$
$$= p_1\ \mathrm{P}_1 + p_2\ \mathrm{P}_2 + \ldots\ldots + p_n\ \mathrm{P}_n. \qquad (1)$$

In Δ, replace the rth row by $a_m, b_m, \ldots\ldots s_m$.

This will make the rth and the mth rows identical. The new determinant obtained will have the value zero.

Expanding it with respect to the new rth row, we have

$$a_m \mathrm{A}_r+b_m \mathrm{B}_r+\ldots+s_m \mathrm{S}_r=0.$$

For a similar reason

$$q_1 \mathrm{P}_1+q_2 \mathrm{P}_2+\ldots+q_n \mathrm{P}_n=0.$$

Thus
$$\begin{aligned} a_m\mathrm{A}_r+b_m\mathrm{B}_r+\ldots+s_m \mathrm{S}_r &= \Delta, \text{ if } m=r, \\ &= 0, \text{ if } m\neq r, \end{aligned} \tag{2}$$

and
$$\begin{aligned} q_1 \mathrm{P}_1+q_2 \mathrm{P}_2+\ldots+q_n \mathrm{P}_n &= \Delta, \text{ if } p=q, \\ &= 0, \text{ if } p\neq q. \end{aligned} \tag{3}$$

2·33. Theorem IV. *If all the elements of a line be multiplied by a common factor, the whole determinant is multiplied by that factor.*

If all the elements of the rth row of Δ are multiplied by m,

$$\begin{aligned} \Delta' &= \{\ ma_r \mathrm{A}_r+mb_r \mathrm{B}_r+mc_r\mathrm{C}_r+\ldots\ldots\} \\ &= m\ \{\ a_r \mathrm{A}_r+b_r \mathrm{B}_r+c_r \mathrm{C}_r+\ldots\ldots\} \\ &= m\Delta. \end{aligned}$$

Cor. *If two parallel lines be such that the elements of one are the equi-multiples of the elements of the other, the determinant vanishes.*

2·34. Theorem V. *If each element of a line consists of m terms, the determinant can be expressed as the sum of m determinants.*

Illustration. Let the determinant be

$$\Delta=\begin{vmatrix} a_1+a'_1 & a_2+a'_2 & a_3+a'_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$=(a_1+a'_1)\ \mathrm{A}_1+(a_2+a'_2)\ \mathrm{A}_2+(a_3+a'_3)\mathrm{A}_3,$$

where $\mathrm{A}_1=b_2c_3-b_3c_2$, $\mathrm{A}_2=b_3c_1-b_1c_3$,

$\mathrm{A}_3=b_1c_2-c_1b_2$.

$$\therefore\ \Delta=(a_1\mathrm{A}_1+a_2\mathrm{A}_2+a_3\mathrm{A}_3)+(a'_1\mathrm{A}_1+a'_2\mathrm{A}_2+a'_3\mathrm{A}_3)$$

$$=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}+\begin{vmatrix} a'_1 & a'_2 & a'_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Here we have to prove that a determinant like

$$\begin{vmatrix} a_1, b_1, \ldots\ldots, p_1' + p_1'' + p_1''' + \ldots\ldots p_1^{(r)} + \ldots\ldots + p_1^{(m)}, \ldots\ldots s_1 \\ a_2, b_2, \ldots\ldots, p_2' + p_2'' + p_2''' + \ldots\ldots p_2^{(r)} + \ldots\ldots + p_2^{(m)}, \ldots\ldots s_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_n, b_n, \ldots\ldots, p_n' + p_n'' + p_n''' \ldots\ldots p_n^{(r)} + \ldots\ldots + p_n^{(m)}, \ldots\ldots s_n \end{vmatrix}$$

$$= \Sigma \begin{vmatrix} a_1, b_1, \ldots\ldots p_1^{(r)}, \ldots\ldots s_1 \\ a_2, b_2, \ldots\ldots p_2^{(r)}, \ldots\ldots s_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_n, b_n, \ldots\ldots p_n^{(r)}, \ldots\ldots s_n \end{vmatrix},$$

where the summation extends from $r=1$ to $r=m$.

Let the co-factors of the elements in the pth column of the determinant on the left be denoted by $P_1, P_2, \ldots\ldots P_n$, then the determinant

$$= [\{p_1' + p_1'' + \ldots + p_1^{(m)}\} P_1 + \{p_2' + p_2'' + \ldots + p_2^{(m)}\} P_2 + \ldots]$$
$$= [\{p_1' P_1 + p_2' P_2 + \ldots + p_n' P_n\} + \{p_1'' P_1 + p_2'' P_2 + \ldots + p_n'' P_n\} + \ldots + \{p_1^{(m)} P_1 + p_2^{(m)} P_2 + \ldots + p_n^{(m)} P_n\}]$$

= the sum of m determinants represented in the Σ notation.

In general, *if each element of one line consists of p terms, of a second parallel line consists of q terms, and of a third consists of r terms, and so on, the determinant can be split up into $pqr\ldots$ determinants.*

2·34. Theorem VI. *If to each element of a line of a determinant be added the equi-multiples of the corresponding elements of one or more parallel lines, the determinant remains unaltered.*

Illustration.

$$\Delta' = \begin{vmatrix} a_1 + b_1 & a_2 + b_2 & a_3 + b_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_3 & c_3 \end{vmatrix}$$

$$=(a_1+b_1)\mathbf{A}_1+(a_2+b_2)\mathbf{A}_2+(a_3+b_3)\mathbf{A}_3$$
$$=(a_1\mathbf{A}_1+a_2\mathbf{A}_2+a_3\mathbf{A}_3)+(b_1\mathbf{A}_1+b_2\mathbf{A}_2+b_3\mathbf{A}_3)$$
$$=(a_1\mathbf{A}_1+a_2\mathbf{A}_2+a_3\mathbf{A}_3)$$
$$=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=\triangle.$$

Proof. Here we have to prove that a determinant like

$$\begin{vmatrix} a_1, b_1,\ldots\ldots,p_1+m'a_1+m''b_1+\ldots\ldots+m(^k)k_1,\ldots\ldots s_1 \\ a_2, b_2,\ldots\ldots,p_2+m'a_2+m''b_2+\ldots\ldots+m(^k)k_2,\ldots\ldots s_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_n, b_n,\ldots\ldots,p_n+m'a_n+m''b_n+\ldots\ldots+m(^k)k_n,\ldots\ldots s_2 \end{vmatrix}$$

$$=\begin{vmatrix} a_1, & b_2,\ldots\ldots p_1\ldots,\ldots s_1 \\ a_2, & b_2\ldots\ldots p_2,\ldots\ldots s_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_n, & b_n,\ldots\ldots p_n,\ldots\ldots s_n \end{vmatrix}=\triangle$$

The determinant on the left is obtained by adding m' times the first, m'' times the second,.. $m(^k)$ times the kth column to the pth column of $\triangle$. In order to prove that this determinant $\triangle'$ is equal to $\triangle$, we notice that it can be broken up into $k+1$ determinants, one of which is $\triangle$, and the other k determinants all vanish in virtue of two columns being identical.

In practice generally, several parallel lines of a determinant may be so altered simultaneously without altering the determinant. As an illustration

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=\begin{vmatrix} a_1+pb_1+qc_1, & b_1+sc_1 & c_1 \\ a_2+pb_2+qc_2, & b_2+sc_2 & c_2 \\ a_3+pb_3+qc_3, & b_3+sc_3 & c_3 \end{vmatrix}.$$

The determinant on the right can be split up into 3×2 determinants, all of which vanish except one, which is equal to $\triangle$.

But care is to be exercised *in using, in the alteration of any line, such lines of* Δ, *as have already been altered.* For, as case (*i*) if we put Δ' as

$$\begin{vmatrix} a_1+pb_1+qc_1 & b_1+ra_1+sc_1 & c_1 \\ a_2+pb_2+qc_2 & b_2+ra_2+sc_2 & c_2 \\ a_3+pb_3+qc_3 & b_3+ra_3+sc_3 & o_3 \end{vmatrix},$$

Δ' will be split up into 3×3, *i.e.*, 9 determinants. Only seven of these will vanish, and the remaining two will be

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } \begin{vmatrix} pb_1 & ra_1 & c_1 \\ pb_2 & ra_2 & c_2 \\ pb_3 & ra_3 & c_3 \end{vmatrix};$$

so here $\Delta'=(1-pr)\ \Delta$.

But as case (*ii*) if we put Δ' as

$$\begin{vmatrix} a_1+pc_1, & b_1+qa_1, & c_1 \\ a_2+pc_2, & b_2+qa_2, & c_2 \\ a_3+pc_3, & b_3+qa_3, & c_3 \end{vmatrix},$$

Δ' will be found to be equal to Δ.

In case (*i*), in altering the *first* column the *second* column has been used and again in altering the *second* column the *first* has been used ; while in case (*ii*), the *second* column has not been used in altering the *first* column, although the *second* has been altered by means of *the altered column* i.e., *first.* So we are led to the following general result :—

If any rth line has been altered by means of sth and tth (say) parallel lines, sth *and* tth *cannot themselves be altered by means of that rth line,* though all the rest parallel lines can be altered by means of the *rth* line (as it originally stood).

The above rule will hold only subject to the condition that one of the parallel lines is left unaltered. For example, in

$$\begin{vmatrix} a_1+pb_1 & b_1+qc_1 & c_1+ra_1 \\ a_2+pb_2 & b_2+qc_2 & c_2+ra_2 \\ a_3+pb_3 & b_3+qc_3 & c_3+ra_3 \end{vmatrix},$$

although 1st column has been altered by means of 2nd, and 2nd by means of 3rd, and 3rd by means of 1st, yet the determinant,

$$=(1+pqr)\ \triangle.$$

It may be noted that alteration by means of *altered lines* means a fresh alteration, and is, therefore, always admissible.

2·35. Theorem VII. *If those elements of a determinant* $\triangle$ *which involve* x, *are polynomials in* x, *and if* $\triangle=0$, *when* a *is substituted for* x, *then* $x-a$ *is a factor of the determinant.*

Since the elements of $\triangle$ involving x are polynomials in x, the the expansion for $\triangle$ will also be a polynomial in x.

As $\triangle$ vanishes for $x=a$, therefore $x-a$ is a factor of $\triangle$.

Exampte. In the determinant

$$\triangle=\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix},$$

if a be substituted for b, first two columns become identical and hence $\triangle$ vanishes. Thus $a-b$ is a factor; similarly $b-c$ and $c-a$ are factors. Since $\triangle$ is *homogeneous and symmetrical* in a, b, c, and of fifth degree, there must be another quadratic factor, symmetrical in a, b, c.

Let

$$\triangle\equiv(a-b)(b-c)(c-a)\{k(a^2+b^2+c^2)+k'(ab+bc+ca)\},\ldots\ldots\ (1)$$

where the numerical factors, k and k', are to be determined.

Putting $a=0$, $b=1$, $c=2$ on the two sides of (1), we see that $4=2\ (5k+2k')$, or $5k+2k'=2$.....(2)

Again, putting $a=0$, $b=2$, $c=3$, in (1),

$$36=6\ (13k+6k'),\ \text{or}\ 13k+6k'=6\ldots\ldots\ldots\ldots(3)$$

Solving (2) and (3), $k=0$ and $k'=1$.

Hence $\triangle=(a-b)\ (b-c)\ (c-a)\ (ab+bc+ca)$.

Though $\triangle$ here may be factorized directly also, without much trouble, the above *illustrates an important process.*

EXAMPLES II (*c*)

1.
$$\begin{vmatrix} 5 & 4 & 0 & -1 \\ 1 & 0 & -5 & 6 \\ 0 & -3 & 2 & 0 \\ -10 & 6 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 0 & -3 & 2 & 0 \\ 5 & 4 & 0 & -1 \\ 1 & 0 & -5 & 6 \\ -10 & 6 & 1 & 4 \end{vmatrix}.$$

[The third row passes over *two* rows to occupy the first place. The expansion can, however, be done by means of the *third row* without any movement.]

$$=3\begin{vmatrix} 5 & 0 & -1 \\ 1 & -5 & 6 \\ -10 & 1 & 4 \end{vmatrix} +2\begin{vmatrix} 5 & 4 & -1 \\ 1 & 0 & 6 \\ -10 & 6 & 4 \end{vmatrix},$$

and expanding the two determinants by means of first row and second row, respectively, the value of the given determinant is obtained to be -1127.

2. $$\Delta=\begin{vmatrix} 43 & 1 & 6 \\ 35 & 7 & 4 \\ 17 & 3 & 2 \end{vmatrix}=\begin{vmatrix} 7.6+1 & 1 & 6 \\ 7.4+7 & 7 & 4 \\ 7.2+3 & 3 & 2 \end{vmatrix}$$

$$=7\begin{vmatrix} 6 & 1 & 6 \\ 4 & 7 & 4 \\ 2 & 3 & 2 \end{vmatrix}+\begin{vmatrix} 1 & 1 & 6 \\ 7 & 7 & 4 \\ 3 & 3 & 2 \end{vmatrix}=0,$$

(since two columns are identical in each of these two determinants).

3. Prove that

$$(i)\ \begin{vmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{vmatrix}=9, \qquad (ii)\ \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}=1.$$

4. Prove that

$$\begin{vmatrix} 1 & 15 & 14 & 4 \\ 12 & 6 & 7 & 9 \\ -4 & 4 & 4 & -4 \\ 12 & -12 & -12 & 12 \end{vmatrix}=0.$$

5. Show that the determinant

$$\begin{vmatrix} 21 & 17 & 7 & 10 \\ 24 & 22 & 6 & 10 \\ 6 & 8 & 2 & 3 \\ 5 & 7 & 1 & 2 \end{vmatrix}$$ vanishes.

[Make two rows identical. Subtract IV row from III. Add I and III and subtract II]

6. Evaluate

$$\begin{vmatrix} 3 & 2 & 1 & 4 \\ 15 & 29 & 2 & 14 \\ 16 & 19 & 3 & 17 \\ 33 & 39 & 8 & 38 \end{vmatrix}.$$ *(P. U. 1940)*

7. Show that

$$\begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+acd \\ 1 & c & c^2 & c^3+abd \\ 1 & d & d^2 & d^3+abc \end{vmatrix} = 0.$$

8. Prove that

$$\begin{vmatrix} x & p & q & 1 \\ a & x & r & 1 \\ a & b & x & 1 \\ a & b & c & 1 \end{vmatrix} = (x-a)(x-b)(x-c).$$

9. Prove that

$$\begin{vmatrix} a & b & c & 0 \\ b & a & 0 & c \\ c & 0 & a & b \\ 0 & c & b & a \end{vmatrix} = \begin{vmatrix} -a & b & c & 0 \\ b & -a & 0 & c \\ c & 0 & -a & b \\ 0 & c & b & -a \end{vmatrix}.$$

Prove that

10.
$$\begin{vmatrix} a & b & c & d \\ -a & b & c & d \\ -a & -b & c & d \\ -a & -b & -c & d \end{vmatrix} = 8\,abcd.$$

11.
$$\begin{vmatrix} b+c & a-c & a-b \\ b-c & c+a & b-a \\ c-b & c-a & a+b \end{vmatrix} = 8\,abc.$$

12.
$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & b+c & a & a \\ 1 & b & c+a & b \\ 1 & c & c & a+b \end{vmatrix} = a^2+b^2+c^2-2bc-2ca-2ab.$$

13. Prove that
$$\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(b+c)(c+a)(a+b).$$

14. Prove the identity
$$\begin{vmatrix} \beta\gamma\delta, & \alpha, & \alpha^2, & \alpha^3 \\ \gamma\delta\alpha, & \beta, & \beta^2, & \beta^3 \\ \delta\alpha\beta, & \gamma, & \gamma^2, & \gamma^3 \\ \alpha\beta\gamma, & \delta, & \delta^2, & \delta^3 \end{vmatrix} = \begin{vmatrix} 1, & \alpha^2, & \alpha^3, & \alpha^4 \\ 1, & \beta^2, & \beta^3, & \beta^4 \\ 1, & \gamma^2, & \gamma^3, & \gamma^4 \\ 1, & \delta^2, & \delta^3, & \delta^4 \end{vmatrix}.$$ [*P. U. 1924*]

[Apply Theorem IV.]

15. Solve the equation
$$\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} = 0.$$ (*P. U. 1939*)

16. Show that the determinant

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$$ vanishes.

[Add the elements of the second column to the corresponding elements of the third, and take out the common factor.]

17. If we denote by capital letters A, B,..., the co-factors of corresponding small letters a, b,...in the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix},$$

show that

$$\Delta' \equiv \begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix} = -(\Sigma A l^2 + 2\Sigma F mn).$$

[Expand Δ *w. r.* to the fourth column, and then each of the three determinants *w. r.* to the third row.]

18. If

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix},$$

show that (with the notation of Ex. 17)

$$BC - F^2 = a\Delta,\ CA - G^2 = b\Delta,\ AB - H^2 = c\Delta,$$
$$GH - AF = f\Delta,\ HF - BG = g\Delta,\ FG - CH = h\Delta.$$

19. Expand the determinant

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & d \end{vmatrix}$$

as a quadratic function of l, m, n. [*P. U. 1926, 1929*]

Prove also that the above determinant is a perfect square, if

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f. & c \end{vmatrix} = 0.$$

20. If ω is one of the complex cube roots of unity, show that

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{vmatrix}$$ vanishes.

21. If $u = ax^4 + 4bx^3 + 6cx^2 + 4dx + c$,

$$u_{11} = ax^2 + 2bx + c$$
$$u_{12} = bx^2 + 2cx + d$$
$$u_{22} = cx^2 + 2dx + e,$$

prove that $$\begin{vmatrix} a & b & c & u_{11} \\ b & c & d & u_{12} \\ c & d & e & u_{22} \\ u_{11} & u_{12} & u_{22} & 0 \end{vmatrix} = -u \begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}.$$

22. Evaluate $$\Delta = \begin{vmatrix} 1 & 10 & 5 & 0 \\ 0 & 1 & 10 & 5 \\ 5 & 10 & 1 & 0 \\ 0 & 5 & 10 & 1 \end{vmatrix}.$$ [*P. U. 1918*]

When, as here, the sum of the elements of each row or each column is the same, we proceed as follows :—

Adding the constituents of each column to the corresponding elements of the first column, we get

$$\Delta = \begin{vmatrix} 16 & 10 & 5 & 0 \\ 16 & 1 & 10 & 5 \\ 16 & 10 & 1 & 0 \\ 16 & 5 & 10 & 1 \end{vmatrix} = 16 \begin{vmatrix} 1 & 10 & 5 & 0 \\ 1 & 1 & 10 & 5 \\ 1 & 10 & 1 & 0 \\ 1 & 5 & 10 & 1 \end{vmatrix} = 16 \begin{vmatrix} 1 & 10 & 5 & 0 \\ 0 & -9 & 5 & 5 \\ 0 & 0 & -4 & 0 \\ 0 & -5 & 5 & 1 \end{vmatrix}.$$

(subtracting *first row* from second, third, and fourth rows).

So $\Delta=16\begin{vmatrix} -9 & 5 & 5 \\ 0 & -4 & 0 \\ -5 & 5 & 1 \end{vmatrix}$. This may most conveniently be expanded by 2nd row ; the sign to be prefixed to the element -4 is obviously positive. Thus $\Delta=16\times(-4)\times 16=-1024$.

23. If $u=ax^2+2bxy+cy^2$, $u'=a'x^2+2b'xy+c'y^3$, prove that

(*i*) $$\begin{vmatrix} y^2 & -xy & x^2 \\ a & b & c \\ a' & b' & c' \end{vmatrix}=\begin{vmatrix} ax+by & bx+cy \\ a'x+b'y & b'x+c'y \end{vmatrix}=-\frac{1}{y}\begin{vmatrix} u & u' \\ ax+by & a'x+b'y \end{vmatrix},$$

(*ii*) $$\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix}=-(ac-b^2)(ax^2+2bxy+cy^2).$$

24. Evaluate

$$\begin{vmatrix} 1 & 15 & 15 & 1 & 0 \\ 0 & 1 & 15 & 15 & 1 \\ 6 & 20 & 6 & 0 & 0 \\ 0 & 6 & 20 & 6 & 0 \\ 0 & 0 & 6 & 20 & 6 \end{vmatrix}.$$ [*P. U. 1919*]

25. Evaluate

$$\begin{vmatrix} 1 & 15 & 14 & 4 \\ 12 & 6 & 7 & 9 \\ 8 & 10 & 11 & 5 \\ 13 & 3 & 2 & 16 \end{vmatrix}.$$ [*P. U. 1925*]

26. Show that

$$\begin{vmatrix} 5 & -10 & 11 & 0 \\ -10 & -11 & 12 & 4 \\ 11 & 12 & -11 & 2 \\ 0 & 4 & 2 & -6 \end{vmatrix}=8100.$$ (*Salmon's Algebre Superieur*)

27. Show that

$$\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2.$$

28. Prove that

$$\begin{vmatrix} b+c & c+a & a+b \\ b'+c' & c'+a' & a'+b' \\ b''+c'' & c''+a'' & a''+b'' \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}.$$

29. Prove that

$$\begin{vmatrix} a & b & c & d \\ a & a+b & a+b+c & a+b+c+d \\ a & 2a+b & 3a+2b+c & 4a+3b+2c+d \\ a & 3a+b & 6a+3b+c & 10a+6b+3c+d \end{vmatrix} = a^4.$$

30. Prove that

$$\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix} = (b-c)(c-a)(a-b)(a+b+c)(a^2+b^2+c^2).$$

31. Show that

$$\begin{vmatrix} (a^2+b^2)/c & c & c \\ a & (b^2+c^2)/a & a \\ b & b & (c^2+a^2)/b \end{vmatrix} = 4abc.$$

32. Show that

$$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+b & a+b \end{vmatrix} = a^3+b^3+c^3-3abc.$$

[$a+b+c$ is easily seen to be a factor.]

33. Factorize $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$.

34. Show that

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3.$$

35. Prove the following identity :—

$$\begin{vmatrix} a+b+nc & (n-1)a & (n-1)b \\ (n-1)c & b+c+na & (n-1)b \\ (n-1)c & (n-1)a & c+a+nb \end{vmatrix} = n(a+b+c)^3.$$

36. Prove the identity :—

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} \equiv (a-b)(a-c)(a-d)(b-c)(b-d)(c-d).$$

37. If $\lambda=\beta\gamma+\alpha\delta$, $\mu=\gamma\alpha+\beta\delta$, $\nu=\alpha\beta+\gamma\delta$, prove that

$$\begin{vmatrix} 1 & 1 & 1 \\ \lambda & \mu & \nu \\ \lambda^2 & \mu^2 & \nu^2 \end{vmatrix} = -(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha-\delta)(\beta-\delta)(\gamma-\delta).$$

Hence show that the following determinant has the same value :—

$$\begin{vmatrix} \beta^2\gamma^2+\alpha^2\delta^2 & \beta\gamma+\alpha\delta & 1 \\ \gamma^2\alpha^2+\delta^2\beta^2 & \gamma\alpha+\beta\delta & 1 \\ \alpha^2\beta^2+\gamma^2\delta^2 & \alpha\beta+\gamma\delta & 1 \end{vmatrix}.$$

38. Show that

$$\begin{vmatrix} (a+b)^2 & ca & bc \\ ca & (b+c)^2 & ab \\ bc & ab & (c+a)^2 \end{vmatrix} = 2abc(a+b+c)^3.$$

39. Show that

$$\Delta = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3.$$

[*P. U. 1923*]

Subtracting the third column from the first and also from the second, and taking out the common factors from the columns thus altered,

$$\Delta = (a+b+c)^2 \begin{vmatrix} b+c-a, & 0, & a^2 \\ 0, & c+a-b, & b^2 \\ c-a-b, & c-a-b, & (a+b)^2 \end{vmatrix}.$$

Again, subtracting from the third row the sum of the first two rows,

$$\Delta = (a+b+c)^2 \begin{vmatrix} b+c-a, & 0, & a^2 \\ 0, & c+a-b, & b^2 \\ -2b, & -2a, & 2ab \end{vmatrix}$$

$$= \frac{(a+b+c)^2}{ab} \begin{vmatrix} a(b+c-a), & 0, & a^2 \\ 0, & b(c+a-b), & b^2 \\ -2ab, & -2ab, & 2ab \end{vmatrix}$$

$$= \frac{(a+b+c)^2}{ab} \times 2ab \begin{vmatrix} a(b+c), & a^2, & a^2 \\ b^2 & b(c+a), & b^2 \\ 0, & 0, & 1 \end{vmatrix}$$

$$= 2(a+b+c)^2 \begin{vmatrix} a(b+c) & a^2 \\ b^2 & b(c+a) \end{vmatrix}$$

$$= 2ab(a+b+c)^2\{(b+c)(c+a)-ab\} = 2abc(a+b+c)^3.$$

40. Verify that

$$\begin{vmatrix} -a(b^2+c^2-a^2) & 2b^3 & 2c^3 \\ 2a^3 & -b(c^2+a^2-b^2) & 2c^3 \\ 2a^3 & 2b^3 & -c(a^2+b^2-c^2) \end{vmatrix} = abc(a^2+b^2+c^2)^3,$$

and evaluate

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}. \qquad [P.U.\ 1927]$$

41. Simplify

$$\begin{vmatrix} x^2 & x & x & 1 \\ ab & a & b & 1 \\ bc & b & c & 1 \\ ca & c & a & 1 \end{vmatrix} = 0$$

into the form $\Sigma\,(x-a)^2(b-c)^2=0$, and deduce that its roots are both imaginary, when a, b, c are all real. [*P.U. 1923*]

Subtracting third column from second, and expanding,

$$x^2\begin{vmatrix} a & b & 1 \\ b & c & 1 \\ c & a & 1 \end{vmatrix} + x\begin{vmatrix} ab & a-b & 1 \\ bc & b-c & 1 \\ ca & c-a & 1 \end{vmatrix} - \begin{vmatrix} ab & a & b \\ bc & b & c \\ ca & c & a \end{vmatrix} = 0.$$

Now the coefficient of $x^2 = -\frac{1}{2}\Sigma(b-c)^2$; the coefficient of x

$$= \begin{vmatrix} a(b-c) & 2a-b-c & 0 \\ -c(a-b) & -(2c-a-b) & 0 \\ ca & c-a & 1 \end{vmatrix} = \Sigma a(b-c)^2.$$

Absolute term $= -\frac{1}{2}\Sigma a^2(b-c)^2$.

Thus the given equation becomes $\Sigma(x-a)^2(b-c)^2=0$.

Now the left-hand side can vanish either if a, b, c are *all* equal, or if the two values of x are imaginary, a, b, c being all

distinct. The first cannot happen as it will turn the equation into an identity, hence the roots are both imaginary.

42. Prove that the determinant

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3.$$

[Trinity, etc.]

43. Prove that

$$\begin{vmatrix} a^2+1 & ab & ac & ad \\ ab & b^2+1 & bc & bd \\ ac & bc & c^2+1 & cd \\ ad & bd & cd & d^2+1 \end{vmatrix}$$

$$= \begin{vmatrix} a^2+1 & b^2 & c^2 & d^2 \\ a^2 & b^2+1 & c^2 & d^2 \\ a^2 & b^2 & c^2+1 & d^2 \\ a^2 & b^2 & c^2 & d^2+1 \end{vmatrix} = a^2+b^2+c^2+d^2+1.$$

2·4. Solution of a system of linear equations.

For the sake of convenience, we shall explain the method with reference to a system of three linear equations in three unknown quantities, but it will be readily seen that the method is perfectly general.

Let the equations be

$$\left.\begin{aligned} a_1x+b_1y+c_1z&=d_1 \\ a_2x+b_2y+c_2z&=d_2 \\ a_3x+b_3y+c_3z&=d_3 \end{aligned}\right\} \quad \ldots(1)$$

Denote the co-factor of a small letter by the corresponding capital letter in the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

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Multiply the equations (1) by A_1, A_2, A_3, respectively, and add; then

$$(a_1A_1+a_2A_2+a_3A_3)x=d_1A_1+d_2A_2+d_3A_3,$$

Using the short notation of Art 2·15, we may write this relation as $x\,(a_1\ b_2\ c_3)=(d_1\ b_2\ c_3)$, where $(d_1\ b_2\ c_3)$ is a determinant obtained from the determinant $(a_1\ b_2\ c_3)$ of the co-efficients, by replacing a_1, a_2, a_3, by d_1 d_2, d_3, respectively.

Hence $$x=\frac{(d_1\ b_2\ c_3)}{(a_1\ b_2\ c_3)}.$$

In the same way, $y=\dfrac{(a_1\ d_2\ c_3)}{(a_1\ b_2\ c_3)}$ and $z=\dfrac{(a_1\ b_2\ d_3)}{(a_1\ b_2\ c_3)}$.

If we take the four simultaneous equations,

$$a_rx+b_ry+c_rz+d_ru=\lambda_r, \qquad (r=1,\ 2,\ 3,\ 4),\ldots(2)$$

the values of x, y, z, u will be given, as above, by

$$x=\frac{(\lambda_1b_2c_3d_4)}{(a_1b_2c_3d_4)},\quad y=\frac{(a_1\lambda_2c_3d_4)}{(a_1b_2c_3d_4)},\quad z=\frac{(a_1b_2\lambda_3d_4)}{(a_1b_2c_3d_4)},\quad \text{and } u=\frac{(a_1b_2c_3\lambda_4)}{(a_1b_2c_3d_4)}.$$

It will be a good exercise for the student to *actually* obtain these results.

2·41. Homogeneous linear equations. If in the equations (2) of the last Art. $\lambda_1=\lambda_2=\lambda_3=\lambda_4=0$, the equations reduce to

$$a_rx+b_ry+c_rz+d_ru=0 \qquad (r=1,\ 2,\ 3,\ 4)\ldots(1)$$

Such a system is called homogeneous. Such were the equations with which we started in Art. 2·13.

Now solving this system of equations by the method of the last Article, we get the determinants in the numerators of the values of x, y, z, u, all zero.

Hence if the determinant $(a_1\ b_2\ c_3\ d_4)$, of the coefficients, is not zero, $x=y=z=u=0$.

Thus we conclude that if $(a_1\ b_2\ c_3\ d_4)\neq0$, the homogeneous equations (1) have no other but the obvious solution

$$x=0=y=z=u.$$

If all the unknown quantities are zero, the system reduces to a nullity. Hence in order that the system of equations be satisfied by values of the unknown quantities, all of which are not zero, it is *necessary and sufficient* that $(a_1\ b_2\ c_3\ d_4)=0$. It is otherwise expressed by the statement, *that the homogeneous equations of system* (1) *are consistent if* $(a_1\ b_2\ c_3\ d_4)=0$. The relation $(a_1\ b_2\ c_3\ d_4)=0$, between the co-efficients of equations (1), must obviously be the result of *eliminating* the four unknown

quantities from the four equations. In general, if *we have n homogeneous linear equations in n quantities, which are simultaneously satisfied for non-zero values of these quantities, the eliminant can at once be written down by equating to zero the determinant formed from the co-efficients in the n equations, taken in order*, as above.

Again, *if the equations* (1) *are consistent, we can find from any three of them the ratios of the four unknown quantities.* For instance, if we take the three equations

$$a_r x + b_r y + c_r z + d_r u = 0. \qquad (r=2, 3, 4).$$

dividing by u and transposing d_r, we get

$$a_r.\frac{x}{u}+b_r.\frac{y}{u}+c_r.\frac{z}{u}=-d_r \qquad (r=2, 3, 4)$$

Now, solving for $\frac{x}{u}$, $\frac{y}{u}$, $\frac{z}{u}$, by the last Article,

$$\frac{x}{u}=\frac{-(d_2b_3c_4)}{(a_2b_3c_4)}, \frac{y}{u}=\frac{-(a_2d_3c_4)}{(a_2b_3c_4)}, \frac{z}{u}=\frac{-(a_2b_3d_4)}{(a_2b_3c_4)} \qquad \ldots(2)$$

Thus the ratios $x : y : z : u$ can be determined. It should be noted that *in the case of homogeneous equations, only proportional values are found because there exists obviously an infinite set of values satisfying such equations.*

If we substitute the values (2) in the fourth equation

$a_1.\frac{x}{u}+b_1.\frac{y}{u}+c_1.\frac{z}{u}+d_1=0$, we get the *eliminant*

$$-a_1(d_2b_3c_4)-b_1(a_2d_3c_4)-c_1(a_2b_3d_4)+d_1(a_2b_3c_4)=0,$$

or $a_1(b_2c_3d_4)-b_1(a_2c_3d_4)+c_1(a_2b_3d_4)-d_1(a_2b_3c_4)=0$, which is the same as $(a_1b_2c_3d_4)=0$.

2·42. From the above it is clear that *in a system of homogeneous equations n equations are sufficient to enable us to eliminate n unknown quantities.* But in a system of *non-homogeneous linear* equations, n equations will *only* give us the *values* of the n unknown quantities, and in order to eliminate these n quantities, we must have one more equation, *i.e.*, $n+1$ equations in all. Thus in order to eliminate *three* unknown quantities from a system *non-homogenoeus linear* equations, the number of such equations must be *four*. If we have the four equations $a_rx+b_ry+c_rz+d_r=0$ $(r=1, 2, 3, 4)$ the values of x, y, z may be found from the three equations

$$a_rx+b_ry+c_rz+d_r=0 \quad (r=2, 3, 4)$$

and substituted in the fourth equation $a_1x+b_1y+c_1z+d_1=0$.

Thus, as above, we shall get the *eliminant* $(a_1b_2c_3d_4)=0$.

2·43. If we have *four* homogeneous equations, involving only *three* unknown quantities, as, for instance,

$$a_r x+b_r y+c_r z=0 \ (r=1, 2, 3, 4),$$

which are all consistent among themselves, each of the following four relations must hold :—

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0, \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_4 & b_4 & c_4 \end{vmatrix}=0, \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}=0, \quad \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}=0.$$

All these four relations are shortly written together as

$$\left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{matrix} \right\|=0,$$

where one *column* is to be omitted in turn.

Similarly, if *four non-homogeneous linear equations* involving only *two* unknown quantities hold simultaneously,

i.e. if $a_r x+b_r y+c_r=0 \ (r=1, 2, 3, 4)$,

the four relations that hold are expressed by the above notation of a rectangular array of elements being bounded by double vertical lines.

Examples II (*d*)

1. Solve the following systems of simultaneous equations :

(*a*) $x+2y+5z=23$, $3x+y+4z=26$, $6x+y+7z=47$.

(*b*) $x+2y+3z=14$, $5x+8y+11z=54$, $3x+5y+z=16$.

(*c*) $x+y+z=1$, $3x+5y+6z=4$, $9x+26y+36z=17$. [*P. U. 1919*]

2. Solve :—

$x+y+z+w=1$,
$3x+4y+25z=-2$,
$4y+7z+w=\frac{4}{5}$,
$7x+7z+10w=4$.

3. Solve :—

$x+y+z=a+b+c$,
$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$,
$ax+by+cz=bc+ca+ab$.
(*P. U. 1917*)

Solve the following equations and put the values of x, y, z, in the simplest form :—

4. $x+y+z=1$,
$ax+by+cz=k$,
$a^2x+b^2y+c^2z=k^2$,

5. $x+y+z=\lambda$,
$\alpha x+\beta y+\gamma z=\lambda^2$,
$a^2x+\beta^2y+\gamma^2z=\lambda^3$.

6. Test the consistency of the equations

$$4x+7y-5z=-3,$$
$$11x-9y+7z=5,$$
$$9x-11y+5z=-1,$$
$$3y+10z=23.$$

7. Show that the equations

$$x+2y+3z+14=0,$$
$$5x+9y+11z+56=0,$$
$$3x+5y+z+16=0,$$
$$2x+7y+5z+30=0,$$

are not consistent with one another. [*P. U. 1920*]

8. Show that the equations

$$x+3y+z=0,$$
$$55x+5y+7z=0,$$
$$9x+7y+3z=0,$$

are consistent; find the ratios $x : y : z$.

9. Show that the following equations

$$5x+7y-13=0,$$
$$-3x+11y-53=0,$$
$$x-5y+23=0,$$
$$2x-3y+18=0,$$

are consistent with one another.

10. Express in determinantal form the conditions, in order that the equations

$$ax+hy+gz=0,$$
$$hx+by+fz=0,$$
$$gx+fy+cz=0,$$
$$lx+my+nz=0,$$

may hold simultaneously.

11. Find λ from the equations

$$ax+hy+g=0,$$
$$hx+by+f=0,$$
$$gx+fy+c=\lambda.$$

12. Show how to find k, where

$$l_1x+m_1y+n_1z+p_1=k,$$
$$l_2x+m_2y+n_2z+p_2=0,$$
$$l_3x+m_3y+n_3z+p_3=0,$$
$$l_4x+m_4y+n_4z+p_4=0,$$

in terms of l's, m's, n's and p's. Carefully establish the principle you make use of. (*P. U. 1926*)

13. Eliminate a, b, c from the equations

$$x=\frac{a}{b-c},\quad y=\frac{b}{c-a},\quad z=\frac{c}{a-b}.$$ [*Math. Tripos*]

2·5. An equation of degree 'n' cannot have more than 'n' roots.

For the sake of definiteness we give a proof for the cubic, the reasoning being general.

If possible, let, α, β, γ, δ be the roots of the equation

$$a_0x^3+a_1x^2+a_2x+a_3=0,$$

so that

$$\begin{cases} a_0\alpha^3+a_1\alpha^2+a_2\alpha+a_3=0, \\ a_0\beta^3+a_1\beta^2+a_2\beta+a_3=0 \\ a_0\gamma^3+a_1\gamma^2+a_2\gamma+a_3=0 \\ a_0\delta^3+a_1\delta^2+a_2\delta+a_3=0. \end{cases}$$

This is a set of four ($n+1$ in general case) equations, linear in a_0, a_1, a_2, a_3. Hence by the theory of linear equations, either

$$a_0=a_1=a_2=a_3=0,$$

or

$$\triangle \equiv \begin{vmatrix} \alpha^3 & \alpha^2 & \alpha & 1 \\ \beta^3 & \beta^2 & \beta & 1 \\ \gamma^3 & \gamma^2 & \gamma & 1 \\ \delta^3 & \delta^2 & \delta & 1 \end{vmatrix} = 0.$$

Now $\triangle$ is the continued product of 6, $[\frac{n}{2}(n+1)$ in the general case], differences $\alpha-\beta$, $\alpha-\gamma$,...of α, β, γ, δ taken in pairs, none of which vanishes, as all the roots are distinct. Thus $\triangle$ cannot be zero; hence

$$a_0=a_1=a_2=a_3.$$

Thus *if an equation of degree 'n' be satisfied by more than 'n' values of x, it must reduce to an identity* (*cf. 1·43*).

Multiplication of Determinants.

2·6. Theorem VIII. *The product of two determinants of any the same order, is itself a determinant of that order.*

We thall give a proof for determinants of the *third* order only, but it will be capable of extension to *determinants of higher orders as well.*

Let $(a_1b_2c_3)$ and $(\alpha_1\beta_2\gamma_3)$ be the two determinants, then we shall prove that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1a_1+b_1\beta_1+c_1\gamma_1, & a_1a_2+b_1\beta_2+c_1\gamma_2, & a_1a_3+b_1\beta_3+c_1\gamma_3 \\ a_2a_1+b_2\beta_1+c_2\gamma_1, & a_2a_2+b_2\beta_2+c_2\gamma_2, & a_2a_3+b_2\beta_3+c_2\gamma_3 \\ a_3a_1+b_3\beta_1+c_3\gamma_1, & a_3a_2+b_3\beta_2+c_3\gamma_2, & a_3a_3+b_3\beta_3+c_3\gamma_3 \end{vmatrix},$$

i.e. $(a_1b_2c_3)(a_1\beta_2\gamma_3)=(d_{11}d_{22}d_{33})$, where $d_{pq}\equiv a_pa_q+b_p\beta_q+c_p\gamma_q$.

Let us consider the three simultaneous equations

$$\left.\begin{aligned} (a_1-\lambda)x+a_2y+a_3z=0 \\ b_1x+(b_2-\lambda)y+b_3z=0 \\ c_1x+c_2y+(c_3-\lambda)z=0 \end{aligned}\right\} \ldots(1),$$

whence by elimination of x, y, z, we get

$$\begin{vmatrix} a_1-\lambda & a_2 & a_3 \\ b_1 & b_2-\lambda & b_3 \\ c_1 & c_2 & c_3-\lambda \end{vmatrix} = 0,$$

or
$$\lambda^3-\lambda^2L+\lambda M-(a_1\ b_2\ c_3) = 0. \qquad (2)$$

If λ_1, λ_2, λ_3 be the roots of this equation,

$$\lambda_1\ \lambda_2\ \lambda_3=(a_1\ b_2\ c_3).$$

Now multiply the equations (1) by a_1, β_1, γ_1, respectively, and add; then multiply (1) by a_2, β_2, γ_2 and add; again, multiply (1) by a_3, β_3, γ_3 and add. The system of equations (1) is then reduced to

$$\begin{aligned} (d_{11}-\lambda a_1)x+(d_{21}-\lambda\beta_1)y+(d_{31}-\lambda\gamma_1)z=0, \\ (d_{12}-\lambda a_2)x+(d_{22}-\lambda\beta_2)y+(d_{32}-\lambda\gamma_2)z=0, \\ (d_{13}-\lambda a_3)x+(d_{23}-\lambda\beta_3)y+(d_{33}-\lambda\gamma_3)z=0, \end{aligned}$$

where $d_{ij}=a_ia_j+b_i\beta_j+c_i\gamma_j$.

By elimination of x, y, z, we get

$$\begin{vmatrix} d_{11}-\lambda a_1 & d_{21}-\lambda\beta_1 & d_{31}-\lambda\gamma_1 \\ d_{12}-\lambda a_2 & d_{22}-\lambda\beta_2 & d_{32}-\lambda\gamma_2 \\ d_{13}-\lambda a_3 & d_{23}-\lambda\beta_3 & d_{33}-\lambda\gamma_3 \end{vmatrix} = 0,$$

or
$$\lambda^3(a_1\ \beta_2\ \gamma_3)-\lambda^2L'+\lambda M'-(d_{11}\ d_{22}\ d_{33})=0.$$

$$\therefore \quad \frac{(d_{11}\ d_{22}\ d_{33})}{(a_1\ \beta_2\ \gamma_3)}=\lambda_1\ \lambda_2\ \lambda_3=(a_1\ b_2\ c_3).$$

Hence
$$(a_1\ b_2\ c_3)\ (a_1\ \beta_2\ \lambda_3)=(d_{11}\ d_{22}\ d_{33}).$$

Examples II (*e*)

1. Multiply

$$\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix} \times \begin{vmatrix} \alpha-i\beta & \gamma-i\delta \\ -\gamma-i\delta & \alpha+i\beta \end{vmatrix},$$

where $i=\sqrt{-1}$, and show that the product-determinant is of the form

$$\begin{vmatrix} \mathrm{D}-i\mathrm{C} & \mathrm{B}-i\mathrm{A} \\ -\mathrm{B}-i\mathrm{A} & \mathrm{D}+i\mathrm{C} \end{vmatrix}.$$

Hence prove that

$$(a^2+b^2+c^2+d^2)(\alpha^2+\beta^2+\gamma^2+\delta^2)=(\mathrm{A}^2+\mathrm{B}^2+\mathrm{C}^2+\mathrm{D}^2),$$

which is *Euler's Theorem that the product of two numbers, each the sum of four squares, is itself the sum of four squares.*

2. Multiply

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix},$$

and establish the identity :—

$$\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} \equiv \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 \equiv (a^3+b^3+c^3-3abc)^2.$$

3. Prove that $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}$

$$\equiv(ax+by+cz)^3+(ay+bz+cx)^3+(az+bx+cy)^3 -3(ax+by+cz)(ay+bz+cx)(az+bx+cy).$$

4. Find the condition, in determinant form, in order that the general expression of the second degree, in two variables, $f(x, y)\equiv ax^2+2hxy+by^2+2gx+2fy+c$, may be resolved into two linear factors.

Let $f(x, y)\equiv(lx+my+n)(l'x+m'y+n')$.

Comparing the coefficients,
$ll'=a$, $mm'=b$, $nn'=c$, $lm'+l'm=2h$, $mn'+m'n=2f$,
and $nl'+n'l=2g$.

Now $\begin{vmatrix} l & l' & 0 \\ m & m' & 0 \\ n & n' & 0 \end{vmatrix}=0$, $\begin{vmatrix} l' & l & 0 \\ m' & m & 0 \\ n' & n & 0 \end{vmatrix}=0$.

Multiplying them we have

$$\begin{vmatrix} 2ll' & lm'+l'm & nl'+n'l \\ ml'+m'l & 2mm' & mn'+m'n \\ nl'+n'l & mn'+m'n & 2nn' \end{vmatrix}=0;$$

hence the condition $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$.

5. Prove that

$$\begin{vmatrix} 2 & \alpha+\beta+\gamma+\delta & \alpha\beta+\gamma\delta \\ \alpha+\beta+\gamma+\delta & 2(\alpha+\beta)(\gamma+\delta) & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) \\ \alpha\beta+\gamma\delta & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) & 2\alpha\beta\gamma\delta \end{vmatrix}=0.$$

(Trinity)

[*Hint.* Multiply two zero-determinants, as in Ex. 4.]

6. Express $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}^2$ as a determinant.

7. Show that

$$\begin{vmatrix} (x-a)^2 & (y-a)^2 & (z-a)^2 \\ (x-b)^2 & (y-b)^2 & (z-b)^2 \\ (x-c)^2 & (y-c)^2 & (z-c)^2 \end{vmatrix}=2\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix}\times\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$$
$$=2(x-y)(y-z)(z-x)(a-b)(b-c)(c-a).$$

8. Find the value of

$$\Delta=\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix},$$

where $l_1^2+m_1^2+n_1^2=1$, etc., and $l_1l_2+m_1m_2+n_1n_2=0$, etc.

(P. U. 1926)

[Square Δ]

9. Show that each of the following determinants is zero :—

$$(i)\begin{vmatrix} a_1\alpha_1+b_1\beta_1 & a_2\alpha_1+b_2\beta_1 & a_3\alpha_1+b_3\beta_1 \\ a_1\alpha_2+b_1\beta_2 & a_2\alpha_2+b_2\beta_2 & a_3\alpha_2+b_3\beta_2 \\ a_1\alpha_3+b_1\beta_3 & a_2\alpha_3+b_2\beta_3 & a_3\alpha_3+b_3\beta_3 \end{vmatrix},$$

$$(ii)\begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix},$$

$$(iii)\begin{vmatrix} 2y_1z_1 & y_1z_2+y_2z_1 & y_1z_3+y_3z_1 \\ y_1z_2+y_2z_1 & 2y_2z_2 & y_2z_3+y_3z_2 \\ y_1z_3+y_3z_1 & y_2z_3+y_3z_2 & 2y_3z_3 \end{vmatrix}.$$

Deter ıinants of special types

2·7. Determinants may be written in such a manner that their elements bear some particular relations with the elements of another determinant, or among themselves. Such determinants possess properties which are of some special and important nature. The classification and study of such determinants will afford great facility in their evaluation or factorization. We shall take up here the more important of such determinants.

2·71. Adjugate Determinants.—In the general determinant $\Delta\equiv(a_1b_2c_3\ldots l_n)$, if the elements be replaced by their co-factors, we get another determinant, $\Delta'\equiv(A_1B_2C_3\ldots L_n)$. This determinant Δ' is called the *Adjugate* of Δ.

To prove that $\Delta' \equiv \Delta^{n-1}$.

Multiplying together the identities

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 & \ldots l_1 \\ a_2 & b_2 & c_2 & \ldots l_2 \\ \ldots\ldots & & & \\ a_n & b_n & c_n & \ldots l_n \end{vmatrix} \text{ and } \Delta' \equiv \begin{vmatrix} A_1 & B_1 & C_1 & \ldots L_1 \\ A_2 & B_2 & C_2 & \ldots L_2 \\ \ldots\ldots & & & \\ A_n & B_n & C_n & \ldots L_n \end{vmatrix},$$

we have, $\Delta\Delta'$

$$\equiv \begin{vmatrix} a_1A_1 + b_1B_1 + \ldots + l_1L_1, \ldots & a_1A_n + b_1B_n + \ldots + l_1L_n \\ a_2A_1 + b_2B_1 + \ldots + l_2L_1, \ldots & a_2A_n + b_2B_n + \ldots + l_2L_n \\ \ldots\ldots & \\ a_nA_1 + b_nB_1 + \ldots + l_nL_1, \ldots & a_nA_n + b_2B_n + \ldots + l_2L_n \end{vmatrix}.$$

Now in this determinant the elements, $a_rA_r + b_rB_r + \ldots + l_rL_r$ $(r=1, 2, 3, \ldots n)$ situated in the principal diagonal are all equal to Δ, while the remaining elements

$$a_r A_s + b_r B_s + \ldots + l_r L_s \text{ (where } r \neq s)$$

all vanish. [See Art. 2·32 Cor.]

Thus we get $\Delta\Delta' \equiv \Delta^n$, or $\Delta' = \Delta^{n-1}$.

2·72. Symmetrical Determinant.—Any two elements of a determinant which are situated in a line perpendicular to the principal diagonal and at equal distances on opposite sides of it, are called **conjugate elements.**

A determinant, in which every two conjugate elements are equal to one another, is called a **Symmetrical Determinant.**

The determinants in Ex. 8, Examples II (*c*), are Symmetrical Determinants.

To prove that the *square of any determinant is a symmetrical determinant.*

Let us take any determinant of the third order and square it.

Thus $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1^2+b_1^2+c_1^2 & a_1a_2+b_1b_2+c_1c_2 & a_1a_3+b_1b_3+c_1c_3 \\ a_2a_1+b_2b_1+c_2c_1 & a_2^2+b_2^2+c_2^2 & a_2a_3+b_2b_3+c_2c_3 \\ a_3a_1+b_3b_1+c_3c_1 & a_3a_2+b_3b_2+c_3c_2 & a_3^2+b_3^2+c_3^2 \end{vmatrix},$$

which is clearly a symmetrical determinant. In the same way, the theorem can be shown to hold for a determinant of any other order.

In the per-symmetrical, or ortho-symmetrical, determinant

$$\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix},$$

all the elements on every line perpendicular to a diagonal are equal and the elements in the principal diagonal are all equal. The factors of such a determinant are found as below :—

By adding together all the other columns to the first, $a+b+c+d$ is seen to be a factor. Again, by adding to the first column the second and subtracting the sum of the third and the fourth, $a+b-c-d$ is seen to be a factor. Similarly $a+c-b-d$ and $a+d-b-c$ are factors. The remaining factor must be numerical, which on comparison of the coefficients of a^4 is seen to be unity. Thus the determinant is equivalent to the product

$$(a+b+c+d)(a+b-c-d)(a+c-b-d)(a+d-b-c).$$

Skew and Skew-Symmetric Determinants

2·73. A determinant in which every element on one side of the diagonal is equal to its conjugate with sign changed, is called a **Skew Determinant.**

Thus

$$\begin{vmatrix} a, & a, & \beta, & \gamma \\ -a, & b, & , & m \\ -\beta, & -l, & c, & n \\ -\gamma, & -m, & -n, & d \end{vmatrix}$$

and the determinant in Ex. 1, Examples II(a), are skew determinants.

If *every element* is equal to its conjugate with sign changed (the diagonal elements being *not excepted* as above), the determinant is called a *Skew-Symmetric determinant.*

Since each element in the principal diagonal, in this case, is its own conjugate, all the elements in the principal diagonal must be zero. Thus every Skew-Symmetric determinant is *zero-axial.* The determinant in Ex. 12, Examples II (b) belongs to this class.

The following two properties of Skew-Symmetric determinants are useful :—

2·731. *A Skew-Symmetrical determinant of odd order vanishes.*
The following method of proof for a determinant of the third order is quite general.

$$\Delta \equiv \begin{vmatrix} 0 & c & b \\ -c & 0 & a \\ -b & -a & 0 \end{vmatrix} = \begin{vmatrix} 0 & -c & -b \\ c & 0 & -a \\ b & a & 0 \end{vmatrix}$$

(the columns have been changed into rows)

$$= (-1)^3 \begin{vmatrix} 0 & c & b \\ -c & 0 & a \\ -b & -a & 0 \end{vmatrix} = -\Delta \; ; \therefore 2\Delta, \text{ or } \Delta = 0.$$

2·732. *A Skew-Symmetric determinant of even order is a perfect square.*
Taking a determinant of the fourth order, we have to prove the identity

$$\begin{vmatrix} 0 & x & y & z \\ -x & 0 & c & b \\ -y & -c & 0 & a \\ -z & -b & -a & 0 \end{vmatrix} = (ax-by+cz)^2.$$

[*P. U. 1922, 1923*]

Adding to a-times the second column, $-b$-times the third, and c-times the fourth, we have

$$\frac{1}{a}\begin{vmatrix} 0 & ax-by+cz & y & z \\ -x & 0 & c & b \\ -y & 0 & 0 & a \\ -z & 0 & -a & 0 \end{vmatrix} = \frac{ax-by+cz}{a} \begin{vmatrix} x & c & b \\ y & 0 & a \\ z & -a & 0 \end{vmatrix}$$

$$= (ax-by+cz)^2.$$

2·74. Circulant Determinants. A determinant like

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix},$$

in which the same elements occur in all the rows and are written in circular order, is called a *Circulant Determinant.* In the example given here the last element of any row is the first element of the next row, so that the same element occurs in the leading diagonal throughout.

Every circulant determinant can be put into factors (real or imaginary), which are equal in number to the order of the determinant.

The following method of factorizing the above determinant is quite general.

The roots of $x^3-1=0$ are $1, \omega, \omega^2$.

Multiplying the second and the third columns by 1 and 1^2, and adding to the first, $a+b+c$ is easily seen to be a factor.

Again, multiplying the second and the third columns by ω, ω^2, respectively, and adding to the first, $a+b\omega+c\omega^2$ is seen to be a factor.

Similarly, $a+b\omega^2+c\omega^4$, or $a+b\omega^2+c\omega$ is a factor.

Thus the determinant

$$=k(a+b+c)\ (a+b\omega+c\omega^2)\ (a+b\omega^2+c\omega),$$

where k must be numerical. On comparing the coefficients of a^3, k is found equal to 1.

The following is an alternative elegant demonstration. We consider a fifth order determinant ; the demonstration is applicable to determinant of any order. Let the determinant be

$$\Delta\equiv\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_5 & a_1 & a_2 & a_3 & a_4 \\ a_4 & a_5 & a_1 & a_2 & a_3 \\ a_3 & a_4 & a_5 & a_1 & a_2 \\ a_2 & a_3 & a_4 & a_5 & a_1 \end{vmatrix}.$$

Write $\phi(\omega)=a_1+a_2\omega+a_3\omega^2+a_4\omega^3+a_5\omega^4$, where $\omega^5=1$.

Let $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5$ be the roots of $x^5=1$, and suppose

$$D\equiv\begin{vmatrix} 1 & \omega_1 & \omega_1^2 & \omega_1^3 & \omega_1^4 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \omega_5 & \omega_5^2 & \omega_5^3 & \omega_5^4 \end{vmatrix},$$

Then $\triangle D=$
$$\begin{vmatrix} \phi(\omega_1) & \phi(\omega_2) & \phi(\omega_3) & \phi(\omega_4) & \phi(\omega_5) \\ \omega_1\phi(\omega_1) & \omega_2\phi(_2) & \omega_3\phi(\omega_3) & \omega_4\phi(\omega_4) & \omega_5\phi(\omega_5) \\ \omega_1^2\phi(\omega_1) & \omega_2^2\phi(\omega_2) & \omega_3^2\phi(\omega_3) & \omega_4^2(\omega_4) & \omega_5^2\phi(\omega_5) \\ \omega_1^3\phi(\omega_1) & \omega_2^3\phi(\omega_2) & \omega_3^3\phi(\omega_3) & \omega_4^3(\omega_4) & \omega_5^3\phi(\omega_5) \\ \omega_1^4\phi(\omega_1) & \omega_2^4\phi(\omega_2) & \omega_3^4\phi(\omega_3) & \omega_4^4(\omega_4) & \omega_{54}\phi(\omega_5) \end{vmatrix}$$
$$=D.\phi(\omega_1)\ \phi(\omega_2)\ \phi(\omega_3)\ \phi(\omega_4)\ \phi(\omega_5).$$
$$\therefore \quad \triangle=\phi(\omega_1)\ \phi(\omega_2)\ \phi(\omega_3)\ \phi(\omega_4)\ \phi(\omega_5).$$

2·8. Sylvester's Dialytic Method of Elimination.

A single unknown quantity may be eliminated from two or more given equations of any degree, in that quantity, by means of determinants. The method may be illustrated by means of a simple example. The method here employed is due to Sylvester and is called the "*Sylvester's Dialytic Method of Elimination*". It is generally applicable when only *two* equations are given, but sometimes, it may be used for more than two equations as well.

Ex. 1. Eliminate x between $ax^3+bx^2+cx+d=0$ ...(1)
and $lx^2+mx+n=0$ ...(2)

Multiplying (1) by x, we have
$$ax^4+bx^3+cx^2+dx+0=0 \quad ...(3)$$
Also (1) may be written as,
$$0.x^4+ax^3+bx^2+cx+d=0 \quad ...(4)$$
From (2) $0.x^4+0.x^3+lx^2+mx+n=0$ ...(5)
Multiplying (2) by x, $0.x^4+lx^3+mx^2+nx+0=0$ ...(6)
,, ,, x^2, $lx^4+mx^3+nx^2+0.x+0=0$ ...(7)

From the five equations, (3), (4), (5), (6), and (7) we eliminate x, x^2, x^3, x^4, and get
$$\begin{vmatrix} a & b & c & d & 0 \\ 0 & a & b & c & d \\ 0 & 0 & l & m & n \\ 0 & l & m & n & 0 \\ l & m & n & 0 & 0 \end{vmatrix}=0.$$

Note.—Elimination can be effected by Bezout's method also :

Let the equations have a root in common.

$$\therefore \quad a\alpha^3+b\alpha^2+c\alpha+\alpha=0,$$

and
$$l\alpha^2+m\alpha+n=0.$$

Hence the following is true for all values of x :—

$$(ax^3+bx^2+cx+d)\,(l\alpha^2+m\alpha+n)$$
$$-(lx^2+mx+n)\,(a\alpha^3+b\alpha^2+c\alpha+d)=0.$$

Cancelling out the factor $x-\alpha$, and collecting the terms we have

$$(al\alpha^2+am\alpha+an)x^2+[am\alpha^2+(an+bm-d)\alpha+bn-dl]x$$
$$+an\alpha^2+(bn-dl)\alpha+cn-md=0.$$

Now, as this is true for all values of x, the coefficients of the various powers of x must severally vanish.

$$\therefore \quad l\alpha^2+m\alpha+n=0,$$
$$am\alpha^2+(an+bm-d)\alpha+bn-dl=0,$$
$$an\alpha^2+(bn-dl)\alpha+cn-md=0.$$

Eliminating α^2, α we get

$$\begin{vmatrix} l & m & n \\ am & an+bm-d & bn-dl \\ an & bn-dl & cn-md \end{vmatrix}=0.$$

Examples II (f)

1. Show that the determinant

$$\begin{vmatrix} 0 & -x & y & -z & u \\ x & 0 & -r & q & n \\ -y & r & 0 & p & -m \\ z & -q & -p & 0 & l \\ -u & -n & m & -l & 0 \end{vmatrix}=0.$$

2. Factorize

$$\begin{vmatrix} x & a & a & a & a \\ a & x & a & a & a \\ a & a & x & a & a \\ a & a & a & x & a \\ a & a & a & & ax \end{vmatrix}.$$

3. Show that

$$\Delta\Delta' \equiv \begin{vmatrix} -a & b & c & d \\ b & -a & d & c \\ c & d & -a & b \\ d & c & b & -a \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{vmatrix}$$

$$= (b+c+d-a)(a-b+c+d)(a+b-c+d)(a+b+c-d)\,\Delta' ;$$

hence find the factors of the first determinant.

Find the factors, real or imaginary, of the determinants :—

4. $\begin{vmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{vmatrix}$,

5. $\begin{vmatrix} 0 & x & y & z \\ x & 0 & z & y \\ y & z & 0 & x \\ z & y & x & 0 \end{vmatrix}$.

6. Show that the result of eliminating x from

$$ax^2+bx+c=0,$$
$$\alpha x^2+\beta x+\gamma=0,$$

is $\begin{vmatrix} a & b & c & 0 \\ 0 & a & b & c \\ \alpha & \beta & \gamma & 0 \\ 0 & \alpha & \beta & \gamma \end{vmatrix} = 0.$ [*P. U. 1917, 1921*]

Examples II (g)

1. Show that

$$\begin{vmatrix} 9 & 13 & 17 & 4 \\ 18 & 28 & 33 & 8 \\ 30 & 40 & 54 & 13 \\ 24 & 37 & 46 & 11 \end{vmatrix} = -15.$$

(*Salmon's Algebre Superieur*)

2. Show that

$$\begin{vmatrix} 7 & -2 & 0 & 5 \\ -2 & 6 & -2 & 2 \\ 0 & -2 & 5 & 3 \\ 5 & 2 & 3 & 4 \end{vmatrix} = -972. \qquad (Salmon)$$

3. Show that the equation

$$\begin{vmatrix} a^3 & b^3 & c^3 \\ (a+x)^3 & (b+x)^3 & (c+x)^3 \\ (2a+x)^3 & (2b+x)^3 & (2c+x)^3 \end{vmatrix} = 0,$$

can be written as $(a+b+c)x^2+3(ab+bc+ca)x+6abc=0$.

4. Solve the equation

$$\begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 6x+2 & 9x+3 & 12x \\ 8x+1 & 12x & 16x+2 \end{vmatrix} = 0.$$

5. If

$$\begin{vmatrix} x+a & b & c & d \\ a & x+b & c & d \\ a & b & x+c & d \\ a & b & c & x+d \end{vmatrix} = 0,$$

prove that

$$x=0, \text{ or } x+a+b+c+d=0.$$

6. Solve

$$\begin{vmatrix} 1-x & 2 & 3 & 4 \\ 1 & 2-x & 3 & 4 \\ 1 & 2 & 3-x & 4 \\ 1 & 2 & 3 & 4-x \end{vmatrix} = 0.$$

7. Show that

$$\begin{vmatrix} x & a_1 & a_2 & a_3 & 1 \\ a_1 & x & a_2 & a_3 & 1 \\ a_1 & a_2 & x & a_3 & 1 \\ a_1 & a_2 & a_3 & x & 1 \\ a_1 & a_2 & a_3 & a_4 & 1 \end{vmatrix} = (x-a_1)(x-a_2)(x-a_3)(x-a_4).$$

8. If ω is one of the imaginary cube roots of unity, show that

$$\begin{vmatrix} 1 & \omega & \omega^2 & \omega^3 \\ \omega & \omega^2 & \omega^3 & 1 \\ \omega^2 & \omega^3 & 1 & \omega \\ \omega^3 & 1 & \omega & \omega^2 \end{vmatrix}^2 = \begin{vmatrix} 1 & 1 & -2 & 1 \\ 1 & 1 & 1 & -2 \\ -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \end{vmatrix}.$$

9. Show that

$$\begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix} \begin{vmatrix} a^2+\lambda^2 & ab+\lambda c & ac-\lambda b \\ ab-\lambda c & b^2+\lambda^2 & bc+\lambda a \\ ac+\lambda b & bc-\lambda a & c^2+\lambda^2 \end{vmatrix} = \lambda^3(\lambda^2+a^2+b^2+c^2)^3.$$

10. Show that

$$\begin{vmatrix} a^2+\lambda & ab & ac & ad \\ ab & b^2+\lambda & bc & bd \\ ac & bc & c^2+\lambda & cd \\ ad & bd & cd & d^2+\lambda \end{vmatrix}$$

is divisible by λ^3, and find the other factor. (Corpus)

11. Prove that

$$\begin{vmatrix} a^3 & 3a^2 & 3a & 1 \\ a^2 & a^2+2a & 2a+1 & 1 \\ a & 2a+1 & a+2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix} = (a-1)^6.$$

(Balliol Ox.)

12. Show that

$$\delta_6 \equiv \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1+a_1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1+a_2 & 1 & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & 1 & 1 & 1+a_5 \end{vmatrix} = a_1a_2a_3a_4a_5.$$

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Deduce $\Delta_5 \equiv \begin{vmatrix} 1+a_1 & 1 & 1 & 1 & 1 \\ 1 & 1+a_2 & 1 & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & 1 & 1+a_5 \end{vmatrix}$

$$= a_1 a_2 a_3 a_4 a_5 \left(1 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5}\right).$$

[The first part can be proved by subtracting the first column from all others. To prove the second we notice that $\Delta_5 = \delta_5 + a_5 \Delta_4$. This can also be proved by subtracting the first column from each of the others.]

13. If $f(x) = (a_1 - x)(a_2 - x)(a_3 - x)(a_4 - x)(a_5 - x)$, show that

$$\begin{vmatrix} a_1 & x & x & x & x \\ x & a_2 & x & x & x \\ x & x & a_3 & x & x \\ x & x & x & a_4 & x \\ x & x & x & x & a_5 \end{vmatrix} = f(x) - x\frac{df}{dx}.$$

[*Hint.* Reduce it to 12 (*ii*)]

14. Show that

$$\begin{vmatrix} x & a & a & \cdots \\ a & x & a & \cdots \\ a & a & x & a \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a & a & \cdots & x \end{vmatrix} = (x-a)^{n-1}[x + \overline{n-1}a].$$

nth order

15. Prove (by multiplication of two determinants, or otherwise)

$$(ax^2 + by^2 + cz^2 - abc - 2xyz)^2 + AX^2 + BY^2 + CZ^2 - ABC - 2XYZ = 0,$$

where $A \equiv y^2 + z^2 + a^2$, $X \equiv yz + x(b+c)$,
$B \equiv z^2 + x^2 + b^2$, $Y \equiv zx + y(c+a)$,
$C \equiv x^2 + y^2 + c^2$, $Z \equiv xy + z(a+b)$.

[*Hint.* $\begin{vmatrix} a & z & y \\ z & b & x \\ y & x & c \end{vmatrix} = abc + 2xyz - ax^2 - by^2 - cz^2.$]

16. If $p_{rs}=\begin{vmatrix} a_r & b_r \\ a_s & b_s \end{vmatrix}$ and $q_{rs}=\begin{vmatrix} c_r & d_r \\ c_s & d_s \end{vmatrix}$,

prove that

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = p_{12}\,q_{34}+p_{13}\,q_{42}+p_{14}\,q_{23}+p_{34}\,q_{12}+p_{42}\,q_{13}+p_{23}\,q_{14}.$$

This can be seen directly, or as follows : The determinants like p_{12}, q_{34} are called complementary minors. It is obvious that there are $4c_2=6$ pairs of complementary minors. The product like $p_{12}\,q_{34}$ contains four terms and these are terms of the given determinant with the same signs. Thus all the 24 terms of the given determinant appear on the right with the same signs. The sign of a product is $+ve.$ or $-ve.$ according as the product of the principal terms of the factor, regarded as a term of Δ is $+ve.$ or $-ve.$

[This is a useful practical method for expanding a determinant of the fourth order.]

17. Prove that

(*i*) $p_{12}\,p_{34}+p_{13}\,p_{42}+p_{14}\,p_{23}=0$,

(*ii*) $(bc)(ad)+(ca)(bd)+(ab)(cd)=0$. (*Laplace*)

18. Show that

$$\begin{vmatrix} a_1 & b_1 & c_1 & p_1 & q_1 & r_1 \\ a_2 & b_2 & c_2 & p_2 & q_2 & r_2 \\ a_3 & b_3 & c_3 & p_3 & q_3 & r_3 \\ 0 & 0 & 0 & \alpha_1 & \beta_1 & \gamma_1 \\ 0 & 0 & 0 & \alpha_2 & \beta_2 & \gamma_2 \\ 0 & 0 & 0 & \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} a_1 & \beta_1 & \gamma_1 \\ a_2 & \beta_2 & \gamma_2 \\ a_3 & \beta_3 & \gamma_3 \end{vmatrix}.$$

Proof. The expansion of the determinant of the sixth order on the left is obtained by taking one and only one element from each row and from each column; and, therefore, each term of the product $(a_1\ b_2\ c_3)(\alpha_1\ \beta_1\ \gamma_2)$ of the two determinants on the right is a term of the determinant on the left and with the same sign. Again, there will be only 36 terms in the product, and the remaining 6!—36 terms in the left hand determinant are to be accounted for.

These terms will contain the elements p, q, r. Consider, for example, the co-factor of r. In this co-factor R_1 of r_1, three elements must be chosen from the last three rows, and only two of these can be chosen from the last two columns; hence at least one of these three elements must be zero, and therefore, every term of the co-factor R_1 is zero. Similarly, the co-factor of every other p, q, r, is zero. This proves that there are no terms which involve the letters p, q, r.

19. Show that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_2 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_2 & c_1 & -1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & -1 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & -1 \\ 0 & 0 & 0 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 0 & 0 & \beta_1 & \beta_2 & \beta_3 \\ 0 & 0 & 0 & \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ (a\alpha) & (b\alpha) & (c\alpha) & \alpha_1 & \alpha_2 & \alpha_3 \\ (a\beta) & (b\beta) & (c\beta) & \beta_1 & \beta_2 & \beta_3 \\ (a\gamma) & (b\gamma) & (c\gamma) & \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} = \begin{vmatrix} (a\alpha) & (b\alpha) & (c\alpha) \\ (a\beta) & (b\beta) & (c\beta) \\ (a\gamma) & (b\gamma) & (c\gamma) \end{vmatrix},$$

where $(pq) \equiv p_1q_1 + p_2q_2 + p_3q_3$.

This proves the rule for multiplication of two determinants of the third order. Extend the result to determinants of any order.

20. If $(f^2-bc)x+(ch-fg)y+(bg-fh)z=0,$
$(ch-fg)x+(g^2-ca)y+(af-gh)z=0,$
$(bg-hf)x+(af-gh)y+(h^2-ab)z=0;$
show that $abc+2fgh-af^2-bg^2-ch^2=0.$

21. Solve the equations

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0,$$
$$ax+by+cz=a+b+c,$$
$$a^2x+b^2y+c^2z=(a+b+c)^2,$$

given that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\neq 0.$

Examine the case when $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0.$

22. Solve the equation

$$\begin{vmatrix} u+a^2x & w'+abx & v'+acx \\ w'+abx & v+b^2x & u'+bcx \\ v'+acx & u'+bcx & w+c^2x \end{vmatrix}=0.$$

23. By means of a determinant, eliminate x, y, z and w from the equations

$$tx+a(y+z+w)=0,$$
$$ty+b(z+w+x)=0,$$
$$tz+c(w+x+y)=0,$$
$$tw+d(x+y+z)=0,$$

and show that the coefficient of t^2 in the result is

$$-(ab+ac+ad+bc+bd+cd).$$ (*London*)

24. Find the values of t for which the equations

$$(t-1)x+(3t+1)y+2tz=0,$$
$$(t-1)x+(4t-2)y+(t+3)z=0,$$
$$2x+(3t+1)y+3(t-1)z=0,$$

are compatible and find the ratios of $x:y:z$, when t has the smallest of these values. What happens when t has the greatest of these values? (*London*)

25. Find the ratio of

$$\begin{vmatrix} \frac{1}{bc}-\frac{1}{f^2} & -\frac{1}{ch} & \frac{1}{fh} \\ af & -fh & ch \\ \frac{1}{fh} & -\frac{1}{af} & \frac{1}{ab}-\frac{1}{h^2} \end{vmatrix}$$

to the square of

$$\begin{vmatrix} h & a & 0 \\ \frac{1}{h} & \frac{1}{b} & \frac{1}{f} \\ 0 & c & f \end{vmatrix}.$$

26. Determine all sets of solutions (x, y, z) of the equations

$$x+y+z=a+b+c,$$
$$a^2x+b^2y+c^2z=a^3+b^3+c^3,$$
$$a^3x+b^3y+c^3z=a^4+b^4+c^4,$$

where a, b, c, are unequal, distinguishing the cases in which $bc+ca+ab$ is different from, or equal to, zero.

27. Prove that if

$$\frac{a}{l^2}+\frac{b}{m^2}+\frac{c}{n^2}=\frac{a}{x^2}+\frac{b}{y^2}+\frac{c}{z^2}=\frac{al}{x^3}+\frac{bm}{y^3}+\frac{cn}{z^3}=0,$$

where a, b, c are not zero, and no two of $\frac{x}{l}$, $\frac{y}{m}$, $\frac{z}{n}$ are equal; then $\frac{x}{l}+\frac{y}{m}+\frac{z}{n}=0$. [*Peterhouse etc., 1931*]

[*Hint.* Eliminate, a, b, c. Simplify the determinant, put $\frac{x}{l}=\lambda$, $\frac{y}{m}=\mu$, $\frac{z}{n}=\nu$.]

28. Show that

$$\begin{vmatrix} a & h & g & u & l_1 \\ h & b & f & v & m_1 \\ g & f & c & w & n_1 \\ u & v & w & d & p_1 \\ l_2 & m_2 & n_2 & p_2 & k \end{vmatrix} = kD-\Sigma l_1 l_2 A - \Sigma F(m_1n_2+m_2n_1),$$

where A, B, etc. are the *co-factors* of the corresponding small letters in

$$D\equiv\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix}.$$

This is Cauchy's Method of Expansion, and is useful in expanding bordered determinants.

29. Prove that the square of a determinant of even order can be exhibited as a skew-symmetric determinant.

The following proof for the fourth order determinant is perfectly general. Let the determinant be

$$\triangle \equiv \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}, \quad \text{then}$$

$$\triangle^2 = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 & b_1 & -a_1 & d_1 & -c_1 \\ a_2 & b_2 & c_2 & d_2 & b_2 & -a_2 & d_2 & -c_2 \\ a_3 & b_3 & c_3 & d_3 & b_3 & -a_3 & d_3 & -c_3 \\ a_4 & b_4 & c_4 & d_4 & b_4 & -a_4 & d_4 & -c_4 \end{vmatrix} = \begin{vmatrix} 0 & l_{12} & l_{13} & l_{14} \\ l_{21} & 0 & l_{23} & l_{24} \\ l_{31} & l_{32} & 0 & l_{34} \\ l_{41} & l_{42} & l_{42} & 0 \end{vmatrix},$$

where $l_{pq}=a_pb_q-a_qb_p+c_pd_q-c_qd_p$. It is obvious that $l_{pq}=-l_{qp}$.

30. Show that

$$\begin{vmatrix} (a+b+c)^2 & d^2 & d^2 & d^2 \\ a^2 & (b+c+d)^2 & a^2 & a^2 \\ b^2 & b^2 & (c+d+a)^2 & b^2 \\ c^2 & c^2 & c^2 & (d+a+b)^2 \end{vmatrix} = 2(a+b+c+d)^4 \Sigma a^2bc.$$

31. Prove by determinants, or otherwise, that the equations

$$5x+3y+2z=12,$$
$$2x+4y+5z=2,$$
$$39x+43y+45z=c,$$

are incompatible unless $c=74$, and that in that case, the equations are satisfied by $x=2+t$, $y=2-3t$, $z=-2+2t$, where t is any arbitrary quantity. (*London*)

32. Multiply $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ $\begin{vmatrix} c_1 & -2b_1 & a_1 \\ c_2 & -2b_2 & a_2 \\ c_3 & -2b_3 & a_3 \end{vmatrix}$,

Hence show that

$$2(a_1b_2c_3)^2 = \begin{vmatrix} 2\triangle_1 & I_{12} & I_{31} \\ I_{12} & 2\triangle_2 & I_{23} \\ I_{31} & I_{33} & 2\triangle_3 \end{vmatrix},$$

where $\triangle_1=a_1c_1-b_1^2$, $I_{12}=a_1c_2+a_2c_1-2b_1b_2$.

CHAPTER III

TRANSFORMATION OF EQUATIONS

3·1. Let $\alpha_1, \alpha_2 \ldots \alpha_n$ be the roots of the equation

$$f(x)=0 \qquad \ldots \text{(i)},$$

it is required to find an equation whose roots are $\phi(\alpha_1), \phi(\alpha_2) \ldots \phi(\alpha_n)$, where $\phi(x)$ is a given function of x.

The relation between a root x of $f(x)=0$, and a root y of the required equation is

$$y=\phi(x) \qquad \ldots \text{(ii)}$$

Thus the required equation will be found by the elimination of x between (i) and (ii).

In general, whenever a root of the required equation can be expressed as a function of a single root of the given equation $f(x)=0$, we have a relation of the form $\phi(x, y)=0$, and the required equation is found by eliminating x between $f(x)=0$ and $\phi(x, y)=0$.

Examples III (*a*)

1. If α, β, γ, be the roots of $ax^3+3bx^2+3cx+d=0$, find the equation whose roots are $\beta+\gamma, \gamma+\alpha, \alpha+\beta$. (*P. U. 1920*)

Here $\beta+\gamma=(\alpha+\beta+\gamma)-\alpha=-\dfrac{3b}{a}-\alpha.$

Hence we get the relation $y=-x-\dfrac{3b}{a}$ …(1)

Eliminate x between (1) and the given equation.

2. If α, β, γ, be the roots of the cubic $x^3+ax^2+bx+c=0$, find the equation whose roots are $\dfrac{\alpha}{\beta+\gamma}, \dfrac{\beta}{\gamma+\alpha}, \dfrac{\gamma}{\alpha+\beta}$.

$$\left\{\text{Here } \frac{\alpha}{\beta+\gamma}=\frac{\alpha}{\Sigma\alpha-\alpha}=\frac{-\alpha}{a+\alpha}.\right\}$$

3. If a, b, c are the roots of $x^3+qx+r=0$, form the equation whose roots are

$$\frac{b+c}{a^2}, \frac{c+a}{b^2}, \frac{a+b}{c^2}. \qquad (P.\ U.\ 1940)$$

If a, b, c be the roots of the equation $x^3+px^2+qx+r=0$, find the equation whose roots are $a(b+c), b(c+a), c(a+b)$.

4. If α, β, γ, be the roots of the cubic $ax^3+3bx^2+3cx+d=0$, find the equation whose roots are

$$\frac{\alpha}{\beta+\gamma-\alpha}, \quad \frac{\beta}{\gamma+\alpha-\beta}, \quad \frac{\gamma}{\alpha+\beta-\gamma}. \qquad [P.\ U.\ 1926]$$

$$\left\{\text{Here} \quad \frac{\alpha}{\beta+\gamma-\alpha}=\frac{\alpha}{\Sigma\alpha-2\alpha}=\frac{-a\alpha}{3b+2a\alpha}, \text{ etc.}\right\}$$

5. Find the equation whose roots are $\dfrac{\alpha}{-\alpha+\beta+\gamma+\delta}$, etc. where α, β, γ, are the roots of the general biquadratic.

6. If α, β, γ be the roots of the equation $x^3-6x^2+11x-6=0$, find the equation whose roots are $\beta^2+\gamma^2$, $\gamma^2+\alpha^2$, $\alpha^2+\beta^2$. [*P. U. 1941 S.*]

7. If α, β, γ, be the roots of the cubic $x^3-ax^2+bx-c=0$, find the equation whose roots are

$$\alpha+\frac{1}{\beta\gamma}, \quad \beta+\frac{1}{\gamma\alpha}, \quad \gamma+\frac{1}{\alpha\beta}.$$

8. If α, β, γ are the roots of $x^3+qx+r=0$, prove that the equation whose roots are $\dfrac{\beta}{\gamma}+\dfrac{\gamma}{\beta}, \dfrac{\gamma}{\alpha}+\dfrac{\alpha}{\gamma}, \dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}$, is $r^2(y+1)^3+q^3(y+1)+q^3=0$.

9. Solve the equation $x^4-2x^3-13x^2+38x-24=0$, given that one root exceeds the other by 2.

Here the functional relation is $\alpha=\beta+2$.

Since α satisfies the given equation , $\beta+2$ also does it.

$\therefore\ (\beta+2)^4-2(\beta+2)^3-13(\beta+2)^2+38(\beta+2)-24=0,$

or $\beta^4+6\beta^3-\beta^2-6\beta=0,$

or $\beta^3+6\beta^2-\beta-6=0,$

i.e., β satisfies the equation $x^3+6x^2-x-6=0$. ... (1)

But, $\because$ β is a root of the original equation, it also satisfies that equation. Therefore, $x-\beta$ must be a common factor in the polynomial of the given equation and of (1).

By successive divisions, the highest common factor is found to be $x-1$. Therefore, $\beta=1$ and $\alpha=3$.

Dividing the given *biquadratic* by $(x-1)(x-3)$, a *quadratic* is obtained, which gives the remaining roots easily. Thus a *biquadratic* has been depressed to a *quadratic*.

The example is a particular case of the theorem :—

If two roots α and β of the equation $f(x)=0$ are connected by the relation $\beta=\phi(\alpha)$, then α is a common root of $f(x)=0$ and $f[\phi(x)]=0$.

For, β being the root of $f(x)=0$, $f(\beta)=0$, *i.e.*, $f[\phi(\alpha)]=0$, *i.e.*, α is a root of the equation $f[\phi(x)]=0$.

The method fails if $f(x)$ *and* $f[\phi(x)]$ *are identical.*

It is obvious that generally in such cases the equation on division by $(x-\alpha)$ $(x-\beta)$, can be reduced by two dimensions.

10. Solve the equation $x^4-8x^3+21x^2-20x+5=0$, given that the sum of two of the roots is 4.

Here $\alpha+\beta=4$, so $\alpha=4-\beta$. Writing $4-\beta$ for x, it will be found that the new equation obtained in β will be identical with the original. So the method of Ex. 9 fails. It can easily be seen that the left-hand side is identical with $(x^2-3x+1)(x^2-5x+5)$.

If x be replaced by $4-x$, the factor x^2-3x+1 transforms into x^2-5x+5, while x^2-5x+5 transforms into x^2-3x+1. This explains the failure of the method. Such equations can, however, be solved by the method of Ex. 9 by assuming an additional relation $\alpha-\beta=2y$; and since $\alpha+\beta=4$, we get $\alpha=y+2$. Substituting for α, we get the following equation in y,

$$y^4-3y^2+1=0.$$

$$\therefore \quad y^2=\frac{6\pm2\sqrt{5}}{4}, \qquad \text{or } y=\frac{1\pm\sqrt{5}}{2}.$$

Hence α and β are found to be $\frac{3\pm\sqrt{5}}{2}$, $\frac{5\pm\sqrt{5}}{2}$.

It is seen that the equation in y contains only even powers.

11. Find all the roots of the equation $x^4-5x^3+11x^2-13x+6=0$ where the roots are connected by the relation $2\beta+3\alpha=7$. *(P. U. 1941)*

12. Solve the equation $3x^3-10x^2+x+6=0$, when α and β are connected by the relation $\alpha\beta-\alpha-2=0$.

Though β is not given as an explicit function of α, it can easily be so expressed, *i.e.*, $\beta=\frac{2}{\alpha}+1$.

$x-\alpha$ is found to be a common factor between the given cubic and $5x^2+x-6=0$.

13. Find the common roots, if any, of the cubics

$$8x^3+24x^2+24x+9=0 \text{ and } 8x^3+72x^2+96x+45=0.$$

Hence find all the roots.

14. A root α of the equation $3x^3-10x^2+7x+10=0$ is connected with a root α' of the equation $x^3-x^2-17x+65=0$ by the

relation $\alpha\alpha'+\alpha'-\alpha+1=0$. Using this fact, solve the two given equations completely. (*P. U. 1923*)

[$\because$ $\alpha'=\frac{\alpha-1}{\alpha+1}$, $\frac{\alpha-1}{\alpha+1}$ satisfies the second equation, and therefore, $6\alpha^3+22\alpha^2+27\alpha+10=0$.

Thus $x-\alpha$ is a common factor of

$3x^3-10x^2+7x+10$ and $6x^3+22x^2+27x+10$.

Successive divisions give the common factor $3x+2$.]

15. The equation : $4x^5-57x^3+64x^2+108x-144=0$ has two roots which are equal in magnitude and opposite in sign. Solve it completely. (*Selwyn 1928*)

[**Sol.** The relation $\beta=-\alpha$ leads to $4x^2-9=0$, which gives $\alpha=\frac{3}{2}$, $\beta=-\frac{3}{2}$. The depressed equation is $x^3-12x+16=0$, one of whose roots is found by inspection.]

16. The two roots x_1, x_2 of $x^4-7x^3+11x^2-7x+10=0$ are connected by $x_2=2x_1+1$. Solve it.

[**Hint.** Put $z=x_2-1=2x_1$. Substituting $z+1$ for x_2 and $z/2$ for x_1 in the equation, it is found that z satisfies the equations $z^4-3z^3-4z^2-2z+8=0$ and $z^4-14z^3+44z^2-56z+160=0$. The H. C. F. of the two expressions on the left is found to be $z-4$. Hence $4=x_2-1=2x_1$, whence $x_1=2$, $x_2=5$.]

3·2. Special Transformations.

In all the articles which follow α_r $(r=1, 2,\ldots,n)$ will denote the roots of the given equation $f(x)=0$, and β_r $(r=1, 2,\ldots,n)$ the roots of the required equation.

3·21. *To change the sign of roots*, i.e., *to find the equation whose roots are* $-\alpha_1, -\alpha_2,\ldots\ldots$

We have $\beta_r=-\alpha_r$, or $\alpha_r=-\beta_r$. Since α_r is a root of $f(x)=0$, $f(\alpha_r)=0$, or $f(-\beta_r)=0$, *i.e.*, $\beta_r, (r=1, 2,\ldots,n)$ are roots of

$$f(-x)=0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Obviously, (1) gives the following working rule for changing the signs of roots :—In a complete equation (*i.e.*, when all the missing terms are supplied by zero coefficients) *change the sign of every alternate coefficient, beginning with the second term*, and the equation obtainted will have the same roots with opposite signs.

3·22. *To multiply the roots by a given number*, i.e., *to find the equation whose roots are* $m\alpha_1$, $m\alpha_2,\ldots$

Let α_1 be a root of $f(x)=0$, then $f(\alpha_1)=0$.

Now if ma_1 is put for y in $f\left(\frac{y}{m}\right)$, we get $f\left(\frac{ma_1}{m}\right)$, or $f(a_1)$ which vanishes. Thus ma_1 is a root of the new equation

$$f\left(\frac{y}{m}\right)=0.$$

Art. 3·21 is a particular case of Art. 3·22, for, put $m=-1$.

3·221. The use of this transformation is to remove fractional coefficients so as to have the co-efficient of the first term unity. If all the coefficients are integers and the coefficient of x^n is p_0, the coefficient of x^n can be rendered unity by multiplying the roots by p_0. If the coefficients are fractional, we multiply each root by the L. C. M. of the denominator of the fractions. But in practice a number much less than the L. C M. is sufficient for the purpose. The rule to get this number is illustrated in the following examples. (For proof see *Peterson Laurent Théorie des Equations Algébriques.*)

Examples III (b)

1. Transform the equations

(*i*) $7x^6-8x^5+9x^4-x^3-x^2-3x+1=0$,

(*ii*) $x^7-8x^4-3x^2-7x+9=0$,

into those with roots of opposite signs, in each case.

2. Transform the equation $3x^3-2x^2-x+1=0$, into another with intergral coefficients, so that the coefficient of x^3 may be equal to 1.

Multiply the roots of $x^3-\frac{2}{3}x^2-\frac{1}{3}x+\frac{1}{3}=0$ by 3. Thus we get the equation $x^3-2x^2-3x+9=0$, each of whose roots is 3 times that of the given equation.

3. Multiply by 6 the roots of the equation

$$6x^4-7x^3+8x^2-7x+2=0.$$

4. Transform the equation $x^3+\frac{2}{3}x^2-\frac{1}{2}x-3=0$, so as to keep $a_0=1$, and make other coefficients all integral.

Here we shall have to multiply the roots by the smallest number 6, which is the least common multiple of the numbers in the demoninators.

5. Remove the fractional coefficients from the following equation, in such a way that the coefficient of the highest term may be unity :—

$$x^4-\frac{1}{2}x^3+\frac{1}{9}x^2-\frac{1}{8}x+\frac{1}{12}=0.$$

Here, the L. C. M. of 2, 9, 8, 12, is 72. But, if we multiplied the roots by 72, that will mean multiplying the absolute term, $\frac{1}{12}$, by as high a number as 72^4. As the following process will show, so much labour, and then obtaining the equation

in a clumsy form, can be avoided by choosing a much smaller number.

Suppose that number to be m. Then we have

$$x^4-\frac{m}{2}x^3-\frac{m^2}{3^2}x^2-\frac{m^3}{2^3}x+\frac{m^4}{2^2.3}=0,$$

for the transformed equation. It is obvious that as only the higher powers of 2 and 3 occur in later terms, it would suffice to take $m=6$. The following rule will be found helpful in finding m :—

The equation may be written in the form

$$x^4-\frac{x^3}{2^1.3^0}+\frac{x^2}{2^0.3^2}+\frac{x}{2^33^0}+\frac{1}{2^23^1}=0.$$

Take the exponents of 2 in order, (1, 0, 3, 2,) and divide them by 1, 2, 3, 4, respectively. We then get 1, 0, 1, $\frac{1}{2}$. The least integer not less than any quotient is 1.

Hence 2^1 is a factor of the multiplier.

Again, take the exponents of 3 in order, (0, 2, 0, 1). Divide them by 1, 2, 3, 4, and we get 0, 1, 0, $\frac{1}{4}$. The least integer is again 1. Thus 3^1 is also a factor of the multiplier.

Hence the multiplier is $2^1\times3^1$, *i.e.*, 6.

6. Obtain the equation whose roots are $-\frac{1}{3}$ times the roots of $5x^4-6x^3+9x^2+54x-81=0$.

7. Make the coefficient of the highest term unity and the other co-efficients all integral in the equation

$$10x^5-5x^3-2x^2-\frac{7}{125}x+\frac{1}{25}=0.$$

8. Remove fractional co-efficients from the equation

$$\frac{2}{3}x^4-\frac{1}{3}x^3+\frac{1}{16}x-\frac{1}{64}=0,$$

making the coefficient of x^4 unity, and change the signs of the roots. How are the roots of the new equation related to those of the given equation ?

9. Remove fractional co-efficients from the equation

$$x^3-\frac{3}{2}x^2-\frac{1}{2^4}x+\frac{1}{2^5}=0.$$

10. Remove fractional co-efficients from the equation

$$x^4-\frac{5}{6}x^3+\frac{5}{12}x^2-\frac{13}{900}=0.$$

11. Transform the equation $x^3-\frac{1}{2}x+\frac{1}{12}x+\frac{1}{18}=0$ into another with integral coefficients, the coefficient of x^3 being unity.

3·23. Reciprocal Roots. *To transform the equation $f(x)=0$, into another whose roots shall be the reciprocals of the roots of the given equation, $f(x)=0$.*

Let x be a root of $f(x)=0$. If y be a root of the new equation, then $y=\frac{1}{x}$, or $x=\frac{1}{y}$.

We have, therefore, to put $x=\frac{1}{y}$, so that $y=\frac{1}{x}$, and every value of y in the new equation is the reciprocal of the corresponding value of x in $f(x)=0$. Thus

$$p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_{n-1}x+p_n=0,$$

is transformed into

$$p_ny^n+p_{n-1}y^{n-1}\ldots+p_2y^2+p_1y+p_0=0$$

Examples III (c)

1. Solve the equation $105x^3-142x^2+60x-8=0$, if the roots be in H.P. *(P. U. 1939)*

Hint. Take the reciprocal equation.

2. Solve the equation $40x^4-22x^3-21x^2+2x+1=0$, if the roots are in H.P.

3. The roots of the equation $x^3-3ax^2+3bx-c=0$ are in H.P., show that the mean root is $\frac{c}{b}$.

4. If the roots of $x^3+px^2+qx+r=0$ be in H.P., then
$$2q^3-9pqr+27r^2=0.$$

3·3. Synthetic Division. Before we take up the next transformation it will be worthwhile to explain the method of synthetic division.

We take the general quantic

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots\ldots+a_{n-1}x+a_n,$$

and the divisor $x-h$.

Let $a_0x^n+a_1x^{n-1}+\ldots\ldots+a_n$
$$\equiv(x-h)\,(b_0x^{n-1}+b_1x^{n-2}+\ldots\ldots+b_{n-1})+\mathrm{R}.$$

Now equating the co-efficients of the various powers of x on both sides, we obtain

$$\begin{aligned}
a_0&=b_0 && \therefore\ b_0=a_0,\\
a_1&=b_1-b_0h, && \therefore\ b_1=a_1+b_0h,\\
a_2&=b_2-b_1h. && \therefore\ b_2=a_2+b_1h,\\
&\ldots\ldots\ldots

We exhibit below the short method of division, deduced from the above :—

$$\begin{array}{lllll} a_0 & a_1 & a_2 & a_3 \ldots a_{n-1} & a_n \quad (h \\ & b_0h & b_1h & b_2h \ldots b_{n-2}h & b_{n-1}h \\ \hline b_0 & b_1 & b_2 & b_3 \ldots b_{n-1} & \mathrm{R} \end{array}$$

It is easy to see that $b_1 = a_0h + a_1$,

$$b_2 = a_0h^2 + a_1h + a_2, \ldots$$

$$\mathrm{R} = a_0h^n + a_1h^{n-1} + \ldots + a_n = f(h).$$

Note that we have to write down the coefficients and the absolute term in order, supplying the missing coefficients by zeros.

3·4. To diminish the roots by a given number.

To transform $f(x)=0$ into another equation whose roots shall be less than the roots of the given equation, each by a given number h.

Let x be a root of $f(x)=0$. If y be a root of the new equation, then $y=x-h$, or $x=y+h$.

Thus the required equation is $f(y+h)=0$. When $f(y+h)$ is expanded by Taylor's Theorem in ascending powers of y, we get

$$f(h) + f'(h)y + \frac{f''(h)}{2!}y^2 + \ldots + \frac{f^n(h)}{n!}y^n = 0 \qquad \ldots(1)$$

In practical applications, it is convenient to use the artifice explained below :—

Let the transformed equation in y be supposed

$$\mathrm{A}_0y^n + \mathrm{A}_1y^{n-1} + \mathrm{A}_2y^{n-2} + \ldots + \mathrm{A}_{n-1}y + \mathrm{A}_n = 0 \qquad \ldots(2),$$

where A's are to be determined. Since $y=x-h$, substitution in (2) gives

$$\mathrm{A}_0(x-h)^n + \mathrm{A}_1(x-h)^{n-1} + \mathrm{A}_2(x-h)^{n-2} + \ldots + \mathrm{A}_{n-1}(x-h) + \mathrm{A}_n = 0 \ldots.(3)$$

Now the left-hand side of (3) is identical with $f(x)$, which shows that if $f(x)$ is divided by $x-h$, the quotient is $\mathrm{A}_0(x-h)^{n-1} + \mathrm{A}_1(x-h)^{n-2} + \ldots + \mathrm{A}_{n-1}$, and the remainder, A_n. Again, if this quotient is divided by $x-h$, the second quotient is $\mathrm{A}_0(x-h)^{n-2} + \mathrm{A}_1(x-h)^{n-3} + \ldots + \mathrm{A}_{n-2}$ and the second remainder, A_{n-1}. If this process is continued n times, the nth quotient will be A_0, and the nth remainder, A_1. Thus division of $f(x)$ by $x-h$ successively, gives the successive remainders A_n, A_{n-1}, $\mathrm{A}_{n-2}, \ldots \mathrm{A}_2$, A_1, and the last quotient A_0, which is the same as a_0. Hence the successive remainders determine the values of the coefficients in the required equation.

If h is positive, the roots will be diminished ; but if h is negative, the roots will be increased. Hence to increase the roots by h, division should be performed by the binomial $x+h$.

Examples III (*d*)

1. Find the equation whose roots are less than the roots of the equation $3x^4-2x^3+4x^2-3x+16=0$ by 3.

3	−2	4	−3	16	(3
	9	21	75	216	
	7	25	72	232 = A_4	
	9	48	219		
	16	73	291 = A_3		
	9	75			
	25	148 = A_2			
	9				
	34 = A_1				

Dividing the quartic $3x^4-2x^3+4x^2-3x+16$ by $x-3$, the first quotient is obtained to be $3x^3+7x^2+25x+72$, and the remainder, 232, which is equal to A_4. Again, division of this quotient by $x-3$ gives the second quotient, $3x^2+16x+73$ and the remainder, 291, which is equal to A_3. Dividing again by $x-3$, the quotient, $3x+25$ and the remainder, 148 ($=A_2$) are obtained. Next process of division gives 3 as the quotient and 34 ($=A_1$) as the remainder. This last quotient, $3=A_0$. Thus the transformed equation is

$$3x^4+34x^3+148x^2+291x+232=0.$$

2. Diminish the roots of $x^4+3x^2-4x+9=0$ by 5.

3. Increase the roots of $3x^5-5x^3+7=0$ by 4.

4. Find the equation whose roots are the roots of

$$4x^5-2x^3+7x-3=0,$$

each increased by 2. (*P.U. 1940*)

5. Transform the cubic equation $3x^3-27x^2+5x-9=0$ into another so that the term containing x^2 may be absent.

The sum of the roots here is 9 and in the transformed equation this sum is to be zero ; hence the sum of all the three roots should be diminished by 9, *i.e.*, each root should be diminished by 3.

This transformation is very important in the solution of equations, and is known as "Removal of second term."

6. Diminish the roots of the equation $x^3-3x^2-9x+5=0$ by c, and find c so that (*i*) the second degree term, and (*ii*) first degree term be absent.

7. Remove the second term from the quartic
$$2x^4-16x^3+2x^2-x+5=0.$$

8. Diminish the roots of $x^3-3x^2+5x+9=0$ by 25.

In a case like this, when the number is big and may involve inconvenient multiplications, the process may be performed in stages. As a first stage the roots may be diminished by 20 and then the roots of the reduced equation may be diminished by 5. The process be put in a contracted form as shown below :—

1	−3	5	9	20
	20	340	6900	
	17	345	6909	5
	20	740	6975	
	37	1085	13,884	
	20	310		
	57	1395		
	5	335		
	62	1730		
	5			
	67			
	5			
	72			

After the roots are diminished by 20, the transformed equation is $x^3+57x^2+1085x+6909=0$, whose coefficients are shown separated by the first broken line. Then the reduction is done by means of 5, and thus the finally reduced equation is $x^3+72x^2+1,730x+13,884=0$.

9. Diminish the roots of the cubic $x^3-60x^2+281x-330=0$ by 55. What conclusions about the roots of the given equation can you draw from the transformed equation ?

[We might diminish the roots first by 50 and then by 5.

The absolute term in the transformed equation will be found to be zero, showing that one of the roots is zero. Since the roots of the given equation are greater by 55 than the roots of the transformed equation, one root of the given equation is 55. Also the other roots may now be easily found.]

10. Find the equation whose roots exceed by 2 the roots of the equation $4x^4+32x^3+83x^2+76x+21=0$.

Hence solve the given equation. (*P.U. 1923*)

11. Show that the same transformation removes the second and fourth terms of the equation

$$x^4+16x^3+72x^2+64x-180=0.$$

Hence solve the equation

12. Remove the second term from the equation

$$x^4+20x^3+143x^2+430x+462=0,$$

and solve the equation

3·5. Removal of terms in general :—

As shown in Art. 3·4, when the roots of

$$f(x)\equiv a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots\ldots+a_{n-1}x+a_n=0 \qquad \ldots(1)$$

are diminished by h, the new equation in y becomes

$$f(h)+yf'(h)+\ldots\ldots+\frac{y^{n-1}}{(n-1)!}f^{n-1}(h)+\frac{y^n}{n!}f^n(h)=0$$

or,

$$a_0y^n+(na_0h+a_1)y^{n-1}+\left\{\frac{n(n-1)}{2!}a_0h^2+(n-1)a_1h+a_2\right\}y^{n-2}+\ldots+f(h)=0 \ldots(2)$$

Now if the second term is to be removed, na_0h+a_1 should vanish, or h should be chosen equal to $-a_1/na_0$.

If any other term is to be removed, the corresponding coefficient, which is always a function of h, should be equated to zero. When the equation so obtained is solved, the required value or values of h are found. It is clear that there will be found *two* values of h which can remove the third term, and so on.

It will be profitable to use the notation

$$U_n\equiv a_0x^n+na_1x^{n-1}+\frac{n(n-1)}{2!}a_2x^{n-2}+\ldots+na_{n-1}x+a_n.$$

Thus if $f(x)=U_n$, it will be easily seen that $f'(x)=nU_{n-1}$, $f''(x)=n(n-1)U_{n-2}$, etc.

Now if $f(h)=A_n$, $f'(h)=nA_{n-1}$, etc., so that A's are the values of U's when h is substituted for x.

Equation (1) of Art. 3·4 will give the transformed equation whose roots are diminished by h, as

$$A_n+nA_{n-1}y+\frac{n(n-1)}{2!}A_{n-2}y^2+\ldots+nA_1y^{n-1}+A_0y^n=0,$$

where $A_0=a_0$, $A_1=a_0h+a_1$, $A_2=a_0h^2+2a_1h+a_2$,

$A_3=a_0h^3+3a_1h^2+3a_2h+a_3$, etc.

3·51. The Cubic.—Let the given cubic be

$$a_0x^3+3a_1x^2+3a_2x+a_3=0. \qquad \ldots(1)$$

When the roots are diminished by h, the new equation in y is

$$a_0(y+h)^3+3a_1(y+h)^2+3a_2(y+h)+a_3=0,$$

or, arranging the terms,

$$a_0y^3+3(a_0h+a_1)y^2+3(a_0h^2+2a_1h+a_2)y$$
$$+a_0h^3+3a_1h^2+3a_2h+a_3=0. \qquad \ldots(2)$$

Now the second term can be removed by putting $h=-a_1/a_0$ *i.e.*, if in (1) we put $x=y-(a_1/a_0)$, the resulting equation in y will be wanting in the second term.

Thus the transformed equation (2) becomes

$$y^3+\frac{3H}{a_0^2}y+\frac{G}{a_0^3}=0, \qquad \ldots(3)$$

where $H=a_0a_2-a_1^2$, $G\equiv a_0^2a_3-3a_0a_1a_2+2a_1^3$.

If the roots be multiplied by a_0, so that the new root is

$$z\equiv a_0y\equiv a_0\left(x+\frac{a_1}{a_0}\right)\equiv a_0x+a_1,$$

the equation (3) is further transformed into

$$z^3+3Hz+G=0. \qquad \ldots(4)$$

This result can be obtained by the direct substitution $z=a_0x+a_1$.

Note that if x_1 be a root of (1), and z_1 the corresponding root of (4), then, $z_1=a_0\left(x_1+\frac{a_1}{a_0}\right)=\frac{a_0}{3}(2x_1-x_2-x_3)$. Thus the roots of (4) are $\frac{a_0}{3}(2x_1-x_2-x_3)$, $\frac{a_0}{3}(2x_2-x_3-x_1)$, $\frac{a_0}{3}(2x_3-x_1-x_2)$.

3·52. The Quartic.—The general quartic with binomial co-efficients may be written as

$$f(x)\equiv a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0. \qquad \ldots(1)$$

Putting $y+h$ for x in (1), we get $f(y+h)=0$,

i.e. $f(h)+y\ f'(h)+\frac{y^2}{2!}f''(h)+\frac{y^3}{3!}f'''(h)+\frac{y^4}{4!}f^{iv}(h)=0$,

or $a_0y^4+4(a_0h+a_1)y^3+6(a_0h^2+2a_1h+a_2)y^2$
$+4(a_0h^3+3a_1h^2+3a_2h+a_3)y$
$+(a_0h^4+4a_1h^3+6a_2h^2+4a_3h+a_4)=0. \qquad \ldots(2)$

Now the second term will be absent from equation (2), if h is chosen so that $a_0h+a_1=0$, or $h=-\frac{a_1}{a_0}$. The equation then

becomes

$$y^4+\frac{6H}{a_0^2}y^2+\frac{4G}{a_0^3}y+\frac{a_0^2I-3H^2}{a_0^4}=0, \qquad \ldots(3)$$

where $H=a_0a_2-a_1^2$, $G\equiv a_0^2a_3-3a_0a_1a_2+2a_1^3$,

$$I\equiv a_0a_4-4a_1a_3+3a_2^2.$$

Multiplying the roots by a_0, we reduce (3) further to the form

$$z^4+6Hz^2+4Gz+(a_0^2I-3H^2)=0, \qquad \ldots(4)$$

where $z\equiv a_0x+a_1$, as in the case of the cubic.

Note that if x_1, x_2, x_3, x_4 be the roots of (1), the roots of (4) are

$$\frac{a_0}{4}(3x_1-x_2-x_3-x_4), \qquad \frac{a_0}{4}(3x_2-x_3-x_4-x_1),$$

$$\frac{a_0}{4}(3x_3-x_4-x_1-x_2), \qquad \frac{a_0}{4}(3x_4-x_1-x_2-x_3).$$

It may be pointed out here that there is another function of the coefficients which plays an important part in the solution of the quartic. The function is

$$a_0a_2a_4-a_0a_3^2+2a_1a_2a_3-a_1^2a_4-a_2^3,$$

and is denoted by J. While the sum of the numerical coefficients vanishes in the case of J also, J is very much like I in the respect that the degree of each term is 3 and the sum of the suffixes is 6, *i.e.*, double the degree of the terms.

Again, the following relation which will be found useful later, on, should be verified :—

$$G^2+4H^3\equiv a_0^2(HI-a_0J).$$

Examples III (*e*)

1. Diminish the roots of $a_0x^3+3a_1x^2+3a_2x+a_3=0$ by h, and find the condition in order that the second and third terms may be removed simultaneously.

The transformed equation will be found to be

$$a_0x^3+3(a_0h+a_1)x^2+3(a_0h^2+2a_1h+a_2)x$$
$$+(a_0h^3+3a_1h^2+3a_2h+a_3)=0.$$

Now the second and third terms are removed simultaneously if $a_0h+a_1=0$ and also $a_0h^2+2a_1h+a_2=0$, which give the condition, $a_0a_2-a_1^2=0$, *i.e.* $H=0$, or $\frac{a_0}{a_1}=\frac{a_1}{a_2}$.

2. Transform, if possible, the equation $x^3-12x^2+48x-72=0$ in such a way that the transformed equation may be wanting in both the second and the third terms.

Here $\frac{a_0}{a_1}=-\frac{1}{4}=\frac{a_1}{a_2}$ and, therefore, both second and third terms can be removed by the same substitution, h being $=4$.

The equation whose roots are less by 4 is found to be $y^3-8=0$, which gives $y=2,\ 2\omega,\ 2\omega^2$.

Thus the roots of the given equation are $6,\ 3\pm\sqrt{-3}$. So, whenever, in a cubic, $\frac{a_0}{a_1}=\frac{a_1}{a_2}$, it can be solved in this manner.

3. Solve completely the cubic $x^3+3x^2+3x+28=0$.

4. What substitutions can remove the third term of the quartic $x^4-13x^3+54x^2-89x+48=0$? Find the reduced equations.

5. Find the condition in order that the second and fourth terms of the quartic $a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$ may be removed by the same transformation.

Proceeding as in Ex. 1, the condition will be found to be $a_0^2a_3-3a_0a_1a_2+2a_1^3=0$, *i.e.* $G=0$.

6. Can the same transformation remove both the second and the fourth terms of the quartic

$$x^4+16x^3+83x^2+152x+84=0\ ?$$

If so, solve the equation completely.

The condition of Ex. 5 is satisfied here. The transformed equation will be found to be $y^4-13y^2+36=0$, whose roots are $\pm 2,\ \pm 3$. Hence the roots of the given equation are found.

7. Solve the biquadratic $x^4-8x^3+19x^2-12x+2=0$.

8. Reduce the cubic $2x^3-3x^2+6x-1=0$ to the form

$$z^3+3Hz+G=0.$$

9. Reduce the quartic $2x^4-4x^3+12x^2-8x+1=0$ to the form in which the second term may be wanting.

10. Find the relation connecting the co-efficients of the equation $a_0x^5+5a_1x^4+10a_2x^3+10a_3x^2+5a_4x+a_5=0$, if the transformation which removes the second term removes the fourth term also. (*Dublin*)

3·6 The sum of powers of the roots of an equation.

Let $f(x)\equiv p_0x^n+p_1x^{n-1}+\cdots+p_n=0$,

then $f'(x)\equiv np_0x^{n-1}+(n-1)p_1x^{n-2}+\cdots+p_{n-1}$.

Also $f(x)=p(x-\alpha)(x-\beta)\cdots(x-\rho)$.

if $\alpha, \beta, \ldots\ldots\rho$ are the roots of $f(x)=0$,

$\therefore$ Taking Logarithmic differentiation.

$$\frac{f'(x)}{f(x)}=\frac{1}{x-\alpha}+\frac{1}{x-\beta}+\cdots+\frac{1}{x-\rho},$$

$$\text{or}\quad f'(x)=\frac{f(x)}{x-\alpha}+\frac{f(x)}{x-\beta}+\cdots+\frac{f(x)}{x-\rho}.$$

By synthetic division

$$\frac{f(x)}{x-\alpha}=p_0x^{n-1}+a_1x^{n-2}+a_2x^{n-3}+\cdots+a_rx^{n-r-1}+\cdots+a_{n-1},$$

where $a_1=p_0\alpha+p_1$, $a_2=p_0\alpha^2+p_1\alpha+p_2$,

$a_r=p_0\alpha^r+p_1\alpha^{r-1}+\cdots+p_r$,

$a_{n-1}=p_0\alpha^{n-1}+p_1\alpha^{n-2}+\cdots+p_{n-2}\alpha+p_{n-1}$.

Writing similar expressions for

$\frac{f(x)}{x-\beta}$, $\frac{f(x)}{x-\gamma}$……, and adding we get

$$\begin{aligned}f'(x)=&np_0x^{n-1}+(p_0S_1+np_1)x^{n-2}\\&+(p_0S_2+p_1S_1+np_2)x^{n-3}\\&+(p_0S_3+p_1S_2+p_2S_1+np_3)x^{n-4}+\cdots\\&+(p_0S_r+p_1S_{r-1}+p_2S_{r-2}+\cdots+p_{r-1}S_1+np_r)x^{n-r-1}+\cdots\\&+(p_0S_{n-1}+p_1S_{n-2}+p_2S_{n-3}+\cdots\cdots p_{n-2}S_1+np_{n-1}).\end{aligned}$$

Equating the coefficients on both sides, we get

$(n-1)\ p_1=p_0S_1+np_1$, $\quad(n-2)\ p_2=p_0S_2+p_1S_1+np_2$.

$(n-3)\ p_3=p_0S_3+p_1S_2+p_2S_1+np_3$,

$(n-4)\ p_4=p_0S_4+p_1S_3+p_2S_2+p_3S_1+np_4$,

$(n-r)\ p_r=p_0S_r+p_1S_{r-1}+\cdots+p_{r-1}S_1+np_r$,

$p_{n-1}=p_0S_{n-1}+p_1S_{n-2}+\cdots+p_{n-2}S_1+np_{n-1}$.

$\therefore\quad p_0S_1+p_1=0 \quad\cdots\cdots\cdots

Thus by successive steps, the value of any S_r can be determined from these relations.

The sum of the negative powers of the roots of $f(x)$ will equal the sum of the corresponding positive powers of the roots of

$$x^n f\left(\frac{1}{x}\right)=0.$$

Cor. If all the roots are real, α^2, β^2.....are all positive, so that $S_2 \equiv \frac{1}{p_0^2}(p_1^2-2p_0p_2)>0$, *i.e.* $p_1^2>2p_0p_2$. Thus if $p_1^2 \leqslant 2p_0p_2$, there must be some complex roots. Thus the equation has at least one pair of complex roots, whenever

$$p_1^2<2\,p_0p_2\ ;\ p_1^2=2p_0p_2\ ;\ p_1=0,\ p_2>0\ ;\ p_1=p_2=0.$$

Examples III (f)

1. Find the sum of the fifth powers of the roots of the equation $x^4-7x^2+4x-3=0$. (*London University*)

Here $F'(x)=4x^3-14x+4$, and $S_1=\Sigma\alpha=0$.

$\therefore\ p_0S_2+p_1S_1+2p_2=0,\ p_0S_3+p_1S_2+p_2S_1+3p_3=0$, etc.

$S_2-14=0$, $\therefore\ S_2=14$;

$S_3+12=0$, $\therefore\ S_3=-12$;

$S_4-7S_2+4S_1-12=0$, $\therefore\ S_4=110$;

$S_5-7S_3+4S_2-3S_1=0$, $\therefore\ S_5=-140$.

Hence $S_5=-140$.

2. Find the sum of the fifth powers of the roots of the equation $x^4-x^3-19x^2+49x-30=0$. (*Davison*)

3. Find the sum of the sixth powers of the roots of the equation $x^4-x^3-7x^2+x+6=0$.

4. If α, β, γ, be the roots of the equation $x^3-7x+7=0$, find $\alpha^{-4}+\beta^{-4}+\gamma^{-4}$.

Sol. α^{-1}, β^{-1}, γ^{-1}, are the roots of $7x^3-7x^2+1=0$, so we may find S_4 for this equation.

Now $S_1=1$, $7S_2-7S_1=0$, $\therefore\ S_2=1$;

$7S_3-7S_2+3=0$. $\therefore\ S_3=\frac{4}{7}$;

$7S_4-7S_3+S_1=0$, $\therefore\ S_4=\frac{3}{7}$;

Thus $\Sigma\alpha^{-4}=\frac{3}{7}$.

5. Find $\Sigma\alpha^{-5}$, where α, β, γ are the roots of the equation $x^3+2x^2-3x-1=0$.

6. Find the sum of the ninth powers of the roots of $x^3+3x+9=0$. (*Bernard and Child*)

[**Solution.** $x^3=-3\,(x+3)$,

$\therefore\ x^9=-27\,(x^3+9x^2+27x+27)$.

Writing α, β, γ for x and adding, we have

$$S_9=-27\,(S_3+9S_2+27S_1+81).$$

Now $S_1=0$, $S_2=-6$ and $S_3+3S_1+27=0$.

$\therefore$ $S_3=-27$.

Hence etc.]

7. Calculate S_8 for the equation

$$x^5-5ax-1=0.$$

$[x^8=(5a+x^{-1})^2, \quad \therefore \quad S_8=125a^2+10aS_{-1}+S_{-2}]$

8. If

$$\alpha+\beta+\gamma=6,$$
$$\alpha^2+\beta^2+\gamma^2=14,$$
$$\alpha^3+\beta^3+\gamma^3=36,$$

prove that

$$\alpha^4+\beta^4+\gamma^4=98.$$

9. If α, β, γ are the roots of the equation $x^3+px+q=0$, prove that

$$\frac{\alpha^5+\beta^5+\gamma^5}{5}=\frac{\alpha^3+\beta^3+\gamma^3}{3}\cdot\frac{\alpha^2+\beta^2+\gamma^2}{2}.$$

Symmetric Functions

3·7. Let $f(x_1, x_2, \ldots\ldots x_n)$

be a polynomial in n independent variables x_1, $x_2, \ldots\ldots x_n$. The polynomial is called **symmetric** in or with respect to these variables, *if the interchange of any two variables does not change the polynomial.* For instance,

$$x_1^2x_2+x_1^2x_3+x_2^2x_1+x_2^2x_3+x_3^2x_1+x_3^2x_2+x_1x_2+x_2x_3+x_3x_1 +5(x_1+x_2+x_3)$$

is a symmetric function of x_1, x_2, x_3. A symmetric function is briefly denoted by placing the symbol Σ before a typical term. The first six terms of the above symmetric function are denoted by $\Sigma x_1^2x_2$, the next three by Σx_1x_2 and next three by $5\,\Sigma x_1$. Thus briefly, the symmetric function is denoted by

$$\Sigma x_1^2x_2+\Sigma x_1x_2+5\,\Sigma x_1,$$

and conversely, when the symmetric function is written symbolically, all the terms can be put down.

When the degree of each term in a symmetric function is the same, the symmetric function is said to be **homogeneous**, otherwise **non-homogeneous.** The symmetric function written above is not homogeneous, It is obvious that a non-homogeneous symmetric function can be expressed as the sum of homogeneous symmetric functions. The symmetric functions of Art. 1·61

$$E_1\equiv\Sigma x_1,\ E_2\equiv\Sigma x_1x_2,\ldots E_{n-1}\equiv\Sigma x_1x_2\ldots\ldots x_{n-1}\cdot$$
$$E_n=x_1x_2\ldots\ldots x_n$$

are called **elementary symmetric functions.**

The number of factors in each term of a homogenous symmetric function is called its **weight**, and the highest power in which each root enters the function is called its **order**. Thus every elementary symmetric function is of order one, and E_1, E_2,...are respectively of weights 1, 2, 3,... The order of $\Sigma_1{}^2x_2$ is two and weight three.

Let $x_1, x_2,\ldots\ldots x_n$, be the roots of the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_{n-1}x+p_n=0. \qquad (1)$$

We proceed to *verify* the following important theorem.

The degree in terms of the coefficients $p_1, p_2\ldots p_n$ of the value of any integral symmetric function is equal to the order of the symmetric function.

Consider a function ϕ of the coefficients, *e.g.*

$$\phi\equiv 2p_1{}^2p_3-p_1p_2{}^2+p_2p_3-5p_1p_4+5p_5.$$

Express ϕ in terms of the elementary symmetric functions by the relations of Art. 1·6

Suppose $\phi=\psi\ (x_1, x_2,\ldots x_n)$.

Now each of $p_1, p_2,\ldots$contains x_1 in the first degree ; therefore the highest power of x_1 will be obtained from terms of the highest degree, *i.e.*, $2p_1{}^2p_3-p_1p_2{}^2$, and from these alone. Thus the order of the symmetric function ψ in the roots is three. It will be found that ψ is the symmetric function $\Sigma x_1{}^3x_2{}^2$.

If ψ is an integral symmetric function of order r of the roots of the equation

$$a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n=0, \qquad (2)$$

*then $a_0{}^r\ \psi$ is a homogeneous function of $a_0, a_1,\ldots$*of degree r.

This follows from what has been said above by writing

$$p_1=\frac{a_1}{a_0},\quad p_2=\frac{a_2}{a_0},\quad \ldots, p_n=\frac{a_n}{a_0}.$$

**In the value in terms of the coefficients $p_1, p_2,\ldots$of the symmetric function ψ of the roots of the equation (1), the sum of the suffixes of $p_1, p_2,\ldots$in each term is equal to the weight of the function.*

Let $\psi\ (x_1, x_2,\ldots x_n)$ be a homogeneous symmetric function of weight ω and $\phi\ (p_1, p_2, \ldots p_n)$ its value in terms of $p_1, p_2,\ldots$ We may note that the sum of the suffixes of $p_1, p_2,\ldots$in each term of ϕ is the same. This follows from the fact that ψ is homogeneous.

Multiply the roots of (1) by λ, by writing $\frac{x}{\lambda}$ for x. This will change the equation (1) to

$$x^n+\lambda p_1x^{n-1}+\lambda^2p_2x^{n-2}+\ldots\ldots+\lambda^{n-1}p_{n-1}x+\lambda^np_n=0.$$

The proof of this article may be omitted.

The effect of such a transformation will be to change
(1) $\psi(x_1, x_2, \ldots x_n)$ to $\lambda^w \psi\ (x_1, x_2, \ldots x_n)$,
(2) $p_1, p_2, \ldots p_n$ to $\lambda p_1, \lambda^2 p_2, \lambda^3 p_3, \ldots\ldots$ respectively,
(3) a term $p_1^q p_2^r p_3^s \ldots$ of ϕ to $\lambda^{q+2r+3s\cdots} p_1^q p_2^r p_3^s \ldots\ldots$

$$\therefore\ w = q + 2r + 3s + \ldots\ldots$$

NOTE. From the above discussion it follows the values of a symmetric function of weight $w < n$, for an equation (1) of degree n will be the same as that of an equation of degree w, *viz.*

$$x^w + p_1 x^{w-1} + \ldots\ldots\ldots + p_w = 0.$$

Examples III (*g*)

1. Find the sum of the reciprocals of the roots of

$$5x^5 - 7x^4 + 9x^3 + 11x^2 - 8x + 7 = 0.$$

2. If p and q are the roots of $\dfrac{1}{x+a} + \dfrac{1}{x+b} + \dfrac{1}{x} = 0$ and $a^2 + b^2 = 4ab$, then $p^2 + q^2 = 6pq$.

3. If $\alpha, \beta, \gamma, \delta$, be the roots of the biquadratic equation $a_0 x^4 + 4a_1 x^3 + 6a_2 x^2 + 4a_3 x + a_4 = 0$, find the value of $a_0^2 \Sigma \alpha^2$. Also find $a_0^2 \Sigma \alpha^2 \beta^2 \gamma^2$.

4. Find $\Sigma \alpha^2 \beta$ for the cubic equation $x^3 + ax^2 + bx + c = 0$.

Here $\Sigma \alpha \equiv \alpha + \beta + \gamma = -a$,
$\Sigma \alpha\beta \equiv \alpha\beta + \beta\gamma + \gamma\alpha = b$.
By multiplication, $\Sigma \alpha . \Sigma \alpha\beta \equiv \Sigma \alpha^2 \beta + 3\alpha\beta\gamma = -ab$.

$$\therefore\ \Sigma \alpha^2 \beta = -ab - 3\alpha\beta\gamma = 3c - ab.$$

5. Find $\Sigma \alpha^2 \beta^2$ for the same cubic.

Squaring $\Sigma \alpha\beta \equiv \alpha\beta + \beta\gamma + \gamma\alpha = b$,
we have $\Sigma \alpha^2 \beta^2 + 2\alpha\beta\gamma\ (\alpha + \beta + \gamma) = b^2$.

$$\therefore\ \Sigma \alpha^2 \beta^2 = b^2 - 2ca.$$

6. For the same cubic, find $(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha)$.

7. If α, β, γ be the roots of the cubic

$$a_0 x^3 + 3a_1 x^2 + 3a_2 x + a_3 = 0,$$

find the values of
(*i*) $a_0^2 \Sigma (2\alpha - \beta - \gamma)(2\beta - \gamma - \alpha)$,
(*ii*) $a_0^3 \Sigma (2\alpha - \beta - \gamma)(2\beta - \gamma - \alpha)(2\gamma - \alpha - \beta)$,

8. If x_1, x_2, x_3, x_4 are the roots of the biquadratic

$$a_0 x^4 + 4a_1 x^3 + 6a_2 x^2 + 4a_3 x + a_4 = 0,$$

find the values of the symmetric functions,

$$a_0^2 \Sigma (3x_1 - x_2 - x_3 - x_4)(3x_2 - x_3 - x_4 - x_1),$$
$$a_0^3 \Sigma (3x_1 - x_2 - x_3 - x_4)(3x_2 - x_3 - x_4 - x_1)(3x_3 - x_4 - x_1 - x_2),$$
$$a_0^4 \Sigma (3x_1 - x_2 - x_3 - x_4)(3x_2 - x_3 - x_4 - x_1)(3x_3 - x_4 - x_1 - x_2) \times (3x_4 - x_1 - x_2 - x_3).$$

9. Find for the quartic of Ex. 8, the value of $\Sigma \alpha^3 \beta$.

Hint. Multiply $\Sigma \alpha^2$ and $\Sigma \alpha\beta$.

10. Deduce for the same quartic the values of $a_0^2\Sigma\alpha\beta^2\gamma^2$.

11. If α, β, γ, δ, are the roots of

$$a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0,$$

find the value of

$$(\alpha\beta+\gamma\delta)(\gamma\alpha+\beta\delta)+(\alpha\beta+\gamma\delta)(\beta\gamma+\alpha\delta)+(\beta\gamma+\alpha\delta)(\gamma\alpha+\beta\delta).$$

(P. U. 1928)

Sol. The symmetric function is the same as $\Sigma\alpha^2\beta\gamma$

$$\therefore\ \alpha+\beta+\gamma+\delta=-\frac{4a_1}{a_0}, \quad \ldots \quad (i)$$

and $$\alpha\beta\gamma+\alpha\beta\delta+\beta\gamma\delta+\gamma\delta\alpha=-\frac{4a_3}{a_0}. \quad \ldots \quad (ii)$$

Multiplying together, we have

$$\Sigma\alpha.\Sigma\alpha\beta\gamma, \text{ or } \frac{16a_1a_3}{a_0^2}=\Sigma\alpha^2\beta\gamma+4\alpha\beta\gamma\delta. \quad \ldots \quad (iii)$$

$$\therefore\ \Sigma\alpha^2\beta\gamma=\frac{4(4a_1a_3-a_0a_4)}{a_0^2}.$$

The student should learn how to express a product like $\Sigma\alpha.\Sigma\alpha\beta\gamma$ as the sum of Σ-functions without writing out their actual expansions, since the latter method is laborious, in general. To obtain the type of Σ-function in the product, it suffices to use a single term (called leader) of one factor say α. Then if we use any term of $\Sigma\alpha\beta\gamma$ which contains α, we get a term of $\Sigma\alpha^2\beta\gamma$, while if we use any term not containing α, say $\beta\gamma\delta$, we get a term $\alpha\beta\gamma\delta$. It remains to find the coefficients of the Σ-function $\Sigma\alpha^2\beta\gamma$, and $\alpha\beta\gamma\delta$. To get $\alpha^2\beta\gamma$ we must take the term α from $\Sigma\alpha$ and $\alpha\beta\gamma$ from $\Sigma\alpha\beta\gamma$, so that the coefficient of $\alpha^2\beta\gamma$ is unity. To get $\alpha\beta\gamma\delta$, we should take α, or β, or γ, or δ, from $\Sigma\alpha$ and the complementary factor $\beta\gamma\delta$, etc. from $\Sigma\alpha\beta\delta$. Hence

$$\Sigma\alpha.\Sigma\alpha\beta\gamma=\Sigma\alpha^2\beta\gamma+\alpha\beta\gamma\delta,$$
$$4\times4=12+4.$$

As a check, we have marked under each Σ the number of its terms obtained from the theory of combinations in Algebra, (not by writing out in full the Σ-functions). To get $\Sigma\alpha^2\beta\gamma$ we have to form combination of three out of α, β, γ, δ. This gives rise to terms of the form $\alpha\beta\gamma$. Now any one of the constituents of $\alpha\beta\gamma$ can be squared. So each term $\alpha\beta\gamma$ gives rise to three terms of $\Sigma\alpha^2\beta\gamma$. Thus there are 12 terms in $\Sigma\alpha^2\beta\gamma$.

12. For the same quartic find the value of

$$(\alpha\beta+\gamma\delta)(\gamma\alpha+\beta\delta)(\alpha\delta+\beta\gamma).$$

The symmetric simplifies to $\alpha\beta\gamma\delta\Sigma\alpha^2+\alpha^2\beta^2\gamma^2\delta^2\Sigma\frac{1}{\alpha^2}$.

13. Find the cubic whose roots, x, y, z satisfy the relations

$$x+y+z=a,$$
$$x^2+y^2+z^2=a^2+2,$$
$$x^3+y^3+z^3=a^3.$$

14. If x, y, z are not zero and

$$x^2=y-z,\ y^2=z-x,\ z^2=x-y,$$

show that x, y, z are the roots of a certain cubic. (*P. U. 1919*)

Find the cubic whose roots are x^2, y^2, z^2.

15. Find the sum of the squares of the differences of the roots of the cubic $a_0x^3+3a_1x^2+3a_2x+a_3=0$.

[Here we have to find $\Sigma(\beta-\gamma)^2$, which is equal to $2\Sigma\alpha^2-2\Sigma\alpha\beta$.]

16. Find $\Sigma(\beta-\gamma)^2$ for the biquadratic

$$a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0.$$

17. Find, for the same biquadratic, the value of

$$a_0^2[(\alpha-\beta)^2(\gamma-\delta)^2+(\beta-\gamma)^2(\alpha-\delta)^2+(\gamma-\alpha)^2(\beta-\delta)^2].$$

The given symmetric function is equivalent to

$$a_0^2[2\Sigma\alpha^2\beta^2-2\Sigma\alpha^2\beta\gamma+12\alpha\beta\gamma\delta].$$

The result is found to be $24(a_0a_4-4a_1a_3+3a_2^2)$.

It should be noted that in the results of Exs. 15, 16 and 17, where the symmetric functions involve differences of the roots only, and the equations are with binomial coefficients, the algebraic sum of the numerical coefficients is always zero.

Also whenever the equation is written in the form

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots\ \ldots+a_{n-1}x+a_n=0,$$

with or without binomial coefficients, the sum of the suffixes of each term in the result is equal to the degree of the symmetric function.

These considerations are helpful in the verification of results.

18. If α, β, γ, δ, are the roots of the equation $a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$, find the equation whose roots are

$$\beta\gamma+\alpha\delta,\ \gamma\alpha+\beta\delta,\ \alpha\beta+\gamma\delta.$$

19. If α, β, γ, δ, be the roots of the above quartic, find the value of the symmetric function

$$[(\gamma-\alpha)(\beta-\delta)-(\alpha-\beta)(\gamma-\delta)]\,[(\alpha-\beta)(\gamma-\delta)$$
$$-(\beta-\gamma)(\alpha-\delta)]\,[(\beta-\gamma)(\alpha-\delta)-(\gamma-\alpha)(\beta-\delta)].$$

This is an important symmetric function in the theory of the biquadratic. To put down the symmetric function, take the roots α, β, γ, and their differences in circular order, $\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$. Subtracting δ from each root we have the three differences $\alpha-\delta$, $\beta-\delta$, $\gamma-\delta$. Combine these in pairs; so we have

$$(\beta-\gamma)(\alpha-\delta),\ (\gamma-\alpha)(\beta-\delta),\ (\alpha-\beta)(\gamma-\delta).$$

Taking the differences of these in cyclic order we get the factors of the symmetric function.

To evaluate the symmetric function, we call the roots of the required cubic in Ex. 18 as λ, μ, ν. So $(\beta-\gamma)(\alpha-\delta)=\nu-\mu$, $(\gamma-\alpha)(\beta-\delta)=\lambda-\nu$, $(\alpha-\beta)(\gamma-\delta)=\mu-\lambda$. Thus the symmetric function takes the form

$$(2\lambda-\mu-\nu)(2\mu-\nu-\lambda)(2\nu-\lambda-\mu)$$
$$=\{(3\lambda-(\lambda+\mu+\nu)\}\{(3\mu-(\lambda+\mu+\nu)\}\{3\nu-(\lambda+\mu+\nu)\}$$
$$=27\left(\lambda-\frac{2a_2}{a_0}\right)\left(\mu-\frac{2a_2}{a_0}\right)\left(\nu-\frac{2a_2}{a_0}\right)$$
$$=-432(a_0a_2a_4+2a_1a_2a_3-a_0a_3^2-a_1^2a_4-a_2^3)/a_0^3.$$

This follows from Ex. 18, which gives $(x-\lambda)(x-\mu)(x-\nu)=0$, when we write $\frac{2a_2}{a_0}$ for x in the left-hand member of the equation.

20. If a, b, c, are the roots of the equation $x^3+px^2+qx+r=0$, find the values of

(*i*) $(b+c-3a)(c+a-3b)(a+b-3c)$,

(*ii*) $\left(\frac{1}{a^2}-\frac{1}{bc}\right)\left(\frac{1}{b^2}-\frac{1}{ca}\right)\left(\frac{1}{c^2}-\frac{1}{ab}\right)$. (*P. U. 1925*)

Since $b+c-3a\equiv b+c+a-4a=-p-4a$, etc.

(*i*) becomes $-(p+4a)(p+4b)(p+4c)$, which is easily calculated.

Again, (*ii*) may be written as

$$-abc\left(\frac{1}{abc}-\frac{1}{a^3}\right)\left(\frac{1}{abc}-\frac{1}{b^3}\right)\left(\frac{1}{abc}-\frac{1}{c^3}\right),$$

or,
$$-r\left(\frac{1}{r}+\frac{1}{a^3}\right)\left(\frac{1}{r}+\frac{1}{b^3}\right)\left(\frac{1}{r}+\frac{1}{c^3}\right),$$

which can be calculated, after we have found Σa^3, Σa^3b^3

21. If α, β, γ, δ are the roots of the equation

$$x^4+px^3+qx^2+rx+s=0,$$

find the value of

$$(1+\alpha^2)(1+\beta^2)(1+\gamma^2)(1+\delta^2).$$

Since α, β, γ, δ are the roots of the given equation,

$$(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)=x^4+px^3+qx^2+rx+s.$$

Writing i and $-i$ for x and observing $i^2=-1$,

$$(i-\alpha)(i-\beta)(i-\gamma)(i-\delta)=(1-q+s)-i(p-r),$$
$$(-i-\alpha)(-i-\beta)(-i-\gamma)(-i-\delta)=(1-q+s)+i(p-r).$$

Multiplying the two

$$(1+\alpha^2)(1+\beta^2)(1+\gamma^2)(1+\delta^2)=(1-q+s)^2+(p-r)^2.$$

22. If α, β, γ be the roots of the cubic $ax^3+3bx^2+3cx+d=0$, show that $a^2(1+\alpha^2)(1+\beta^2)(1+\gamma^2)=(a-3c)^2+(3b-d)^2$.

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23. Obtain a cubic equation whose roots are the values of x, y, z, given by

$$x+y+z=3,$$
$$x^2+y^2+z^2=5,$$
$$x^3+y^3+z^3=7.$$

Hence prove that $x^4+y^4+z^4=9$. [*King's College, etc., 1927*]

24. Express $bc+ca+ab$ and abc in terms of s, p and q. where

$$2s=a+b+c,$$
$$2p=a^2+b^2+c^2,$$
$$3q=a^3+b^3+c^3.$$ [*Peterhouse, etc., 1930*]

25. If $a+b+c=0$, show that

(*i*) $a^5+b^5+c^5=5(bc+ca+ab)abc$,

(*ii*) $a^7+b^7+c^7=7abc(bc+ca+ab)^2$.

26. Use the last example to show that

$$(x+a)^5-x^5-a^5=5ax(x+a)(x^2+ax+a^2)$$
$$(x+a)^7-x^7-a^7=7ax(x+a)(x^2+ax+a^2)^2.$$

27. If $\alpha+\beta+\gamma+\delta=0$, show that

$$\alpha^5+\beta^5+\gamma^5+\delta^5=-5\Sigma\alpha\beta\Sigma\alpha\beta\gamma.$$

Hence solve the equation

$$(x+a+b)^5-x^5-a^5-b^5=0.$$

[**Note.** $(x+a+b)+(-x)+(-a)+(-b)=0$.]

28. If α, β, γ are the roots of $x^3+p_1x-p_3=0$ and $S_p=\alpha^p+\beta^p+\gamma^p$, prove that

$$\frac{S_{11}}{11}=\frac{S_3}{3}\cdot\frac{S_8}{2}-\frac{S_3^3}{9}\cdot\frac{S_2}{2}.$$

29. If α, β, γ are the roots of the equation $x^3+qx+r=0$, prove that

$$3(\alpha^2+\beta^2+\gamma^2)(\alpha^5+\beta^5+\gamma^5)=5(\alpha^3+\beta^3+\gamma^3)(\alpha^4+\beta^4+\gamma^4).$$

[*St. John's College Camb.*]

Solution : Here $S_1=0$, and $S_2=-2q$.

Multiply $x^3+qx+r=0$ by x^{n-3}.

$$\therefore\quad x^n+qx^{n-2}+rx^{n-3}=0,$$

Since α, β, γ are the roots of the equation,

$$\therefore\quad \alpha^n+q\alpha^{n-2}+r\alpha^{n-3}=0,$$
$$\beta^n+q\beta^{n-2}+r\beta^{n-3}=0,$$
$$\gamma^n+q\gamma^{n-2}+r\gamma^{n-3}=0.$$

Adding up $S_n+qS_{n-2}+rS_{n-3}=0.$

Put $n=3$, 4, 5 respectively, then

$$S_3+qS_1+rS_0=0, \quad \ldots (i)$$
$$S_4+qS_2+rS_1=0, \quad \ldots (ii)$$
$$S_5+qS_3+rS_2=0. \quad \ldots (iii)$$

$\therefore$ $S_3=-3r$, $\cdot$ ($\because$ $S_0=3$), $S_4=2q^2$, $S_5=5qr$.

Hence $3\Sigma\alpha^2\Sigma\alpha^5=3(-2q)(5qr)=5(-3r)(2q^2)=5\Sigma\alpha^3\Sigma\alpha^4$.

30. If $\alpha+\beta+\gamma=0$, show that

$$\alpha^{n+3}+\beta^{n+3}+\gamma^{n+3}=\alpha\beta\gamma(\alpha^n+\beta^n+\gamma^n)$$
$$+\tfrac{1}{2}(\alpha^2+\beta^2+\gamma^2)(\alpha^{n+1}+\beta^{n+1}+\gamma^{n+1}).$$

[*Caius College*]

[*Hint.* Suppose that α, β, γ, are the roots of a cubic of the form $x^3+qx+r=0$.]

31. Prove that the sum of the twentieth powers of the roots of the equation $x^4+ax+b=0$ is $50a^4b^2-5b^5$. [*Radford*]

[If α be a root, $-\alpha^{20}=(a\alpha+b)^5$, $\therefore$ $-S_{20}=a^5S_5+5a^4bS_4+10a^3b^2S_3+10a^2b^3S_2+5ab^4S_1+4b^5$. Calculate S_1, S_2, S_3, S_4, S_5, and substitute.]

32. Show that $S_r=0$, $[r=2, 3\ldots\ldots, n]$ for the equation

$$x^n+\frac{x^{n-1}}{1\,!}+\frac{x^{n-2}}{2\,!}+\ldots\ldots+\frac{1}{n\,!}=0.$$

33. If α, β, γ be the roots of $x^3+3qx+r=0$, evaluate $\Sigma\alpha^3$, $\Sigma\alpha^{-3}$. Hence show that

$$(\alpha^2\beta+\beta^2\gamma+\gamma^2\alpha)(\alpha\beta^2+\beta\gamma^2+\gamma\alpha^2)=9(3q^3+r^2).$$

3·8. Equations whose roots are the symmetric functions of the roots of a given equation.

In the present section we discuss a simple case for the cubic only, though the method is general.

Equation of squared differences of the cubic.

Let α, β, γ be the roots of the cubic

$$f(y)\equiv y^3+3Hy+G=0.$$

Let $$2\alpha=u+v, \qquad 2\beta=u-v, \tag{A}$$

then $$\alpha+\beta=u, \qquad \alpha-\beta=v.$$

Hence $$0=f(\alpha)=f\left(\frac{u}{2}+\frac{v}{2}\right),$$

or $$u^3+3u^2v+3u(v^2+4H)+v^3+12Hv+8G=0.$$

Also $$0=f(\beta)=f\left(\frac{u}{2}-\frac{v}{2}\right),$$

or $$u^3-3u^2v+3u(v^2+4H)-v^3-12Hv+8G=0.$$

By addition and subtraction, we get

$$u^3+3u(v^2+4H)+8G=0, \tag{i}$$
$$3u^2v+v^3+12Hv=0.$$

Now v being the difference of any two roots, cannot vanish.

Hence $$3u^2+12H+v^2=0. \tag{ii}$$

From 3 times (1) we subtract u times (ii), and get

$$u=\frac{-3G}{v^2+3H}.$$

Substituting this value of u in (i),

$$-27G^3-9G(v^2+4H)(v^2+3H)^2+8G(v^2+3H)^3=0,$$

or $$v^6+18Hv^4+81H^2v^2+108H^3+27G^2=0.$$

Put $z=v^2=(\alpha-\beta)^2$; the required equation is

$$z^3+18Hz^2+81H^2z+27(G^2+4H^3)=0. \qquad \text{(iii)}$$

NOTE 1. Suppose x_1, x_2, x_3 are the roots of the cubic

$$a_0x^3+3a_1x^2+3a_2x+a_3=0. \qquad \textbf{(B)}$$

The substitution $z=a_0x+a_1$ transforms the cubic to **(A)**, and we have

$$a_0x_1+a_1=\alpha,\ a_0x_2+a_1=\beta,\ a_0x_3+a_1=\gamma.$$

$$\therefore\ a_0^2(x_1-x_2)^2=(\alpha-\beta)^2,\ a_0^2(x_2-x_3)^2=(\beta-\gamma)^2,$$

$$a_0^2(x_3-x_1)^2=(\alpha-\gamma)^2.$$

Thus the equation of the squared differences of **(B)** can be deduced from (iii) by the transformation $a_0^2y=z$.

The equation in y will be the required equation.

NOTE 2. Put $z=(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta$

$$=(\Sigma\alpha-\gamma)^2-\frac{4\alpha\beta\gamma}{\gamma} \quad =\gamma^2+\frac{4G}{\gamma}.$$

Hence the required transformation is

$$z=y^2+\frac{4G}{y},\ \text{or}\ y^3-zy+4G=0.$$

To obtain the equation of squared differences, now we eliminate y between

$$y^3+3Hy+G=0 \text{ and } y^3-zy+4G=0.$$

Subtracting, $(3H+z)y-3G=0$, or $y=\dfrac{3G}{z+3H}$.

Substituting this value of y in **(A)**, we get the equation (iii).

NOTE 3. If we eliminate v^2 from (i) and (ii), we get a cubic in $u=\alpha+\beta$. Hence this equation will give the sum of the roots of the equation **(A)** in pairs.

3·81. The equation (iii) enables us to determine the nature of the roots of the equation **(A)**, and therefore of **(B)** also.

If α, β, γ are the roots of **(A)**, then

$$(\alpha-\beta)^2,\ (\beta-\gamma)^2,\ (\gamma-\alpha)^2$$

are the roots of (iii).

$$\therefore\ (\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2=-27\ (G^2+4H^3).$$

(1) If α, β, γ are all real, each expression on the left is positive, therefore

$$G^2+4H^3<0\,.$$

(2) If two of the roots are imaginary, say

$$\beta=a+ib,\ \gamma=a-ib,\ \text{then}\ \alpha=-2a,$$

a and b being real,

$$\text{and } (\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2=(-3a-ib)^2(2ib)^2(3a-ib)^2$$
$$=-4b^2(9a^2+b^2)^2=-27(G^2+4H^3).$$

Hence $$G^2+4H^3>0.$$

(3) If two of the roots are equal, one of the factors on the left is zero,

$$\therefore\ G^2+4H^3=0.$$

(3i) If the equation (**A**) has all the three roots equal, the equation (iii) has all zero roots,

$$\therefore\ H=0,\quad G^2+4H^3=0\,;$$

or $$H=0 \quad \text{and} \quad G=0.$$

Conversely,

The equation has real, equal or complex roots according as

$$G^2+4H^3\gtreqless 0.$$

This follows from the fact, that the three cases enumerated above are mutually exclusive.

Examples III (h)

1. Find the equation of squared differences of the cubic $2x^3-3x^2-x+1=0$, showing all the necessary steps.

2. Find the equation of squared differences of the cubic

$$x^3+qx+r=0.$$

3. Find the equation of squared differences of the cubic $2x^3-3x^2+3x-2=0$. Verify by actually solving the cubic and then framing the equation of squared-differences.

4. Find the equation of squared differences of the cubic

$$3x^3-11x^2+8x+4=0.$$

What conclusion can you draw about the nature of the roots from the transformed equation ?

Examples III (i)

1. Show that the condition that the cubic equation

$$x^3-px^2+qx-r=0,$$

should have a pair of roots of the form $\alpha\pm i\alpha$ is

$$(p^2-2q)(q^2-2pr)-r^2=0.$$

Solve the equation $x^3-7x^2+20x-24=0$.

If we call the three roots a, b, c ;

$$a^2+b^2+c^2=(\alpha+i\alpha)^2+(\alpha-i\alpha)^2+c^2=c^2.$$

Also $$\Sigma a^2=(\Sigma a)^2-2\Sigma ab=p^2-2q.$$

$$\therefore\quad c^2=p^2-2q,$$

and since c is a root of the given equation, we get the required condition after substitution and simplification.

2. If $\alpha+i\beta$ be a root of the equation

$$x^3+px+r=0,$$

prove that 2α is a root of

$$x^3+px-r=0.$$

3. If α, β, γ be the roots of the equation $ax^3+bx^2+cx+d=0$, show that the equation whose roots are α^2, β^2, γ^2, is

$$a^2y^3+y^2(2ac-b^2)+y(c^2-2bd)-d^2=0.$$

4. If α, β, γ be the roots of the equation $ax^3+bx^2+cx+d=0$, form the equation whose roots are $\alpha^2+\beta^2$, $\beta^2+\gamma^2$, $\gamma^2+\alpha^2$. *(P. U. 1921)*

[*Hint.* $\beta^2+\gamma^2=\Sigma\alpha^2-\alpha^2=\dfrac{b^2-2ac}{a^2}-\alpha^2.$

Hence the transformation is $z=\dfrac{b^2-2ac}{a^2}-x^2.$

Or. The transformation between z and y of Ex. 3 is

$$z=\frac{b^2-2ac}{a^2}-y].$$

5. If α, β, γ are the roots of the cubic $x^3-ax^2+bx-c=0$, find the equation whose roots are the three values of $\alpha\beta+\alpha\gamma$.

6. If α, β, γ, δ are the roots of the equation $a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$, show that $a_0^2a_3+2a_1^3-3a_0a_1a_2=0$, if $\alpha+\beta=\gamma+\delta$.

7. If α, β, γ, δ are the roots of the quartic of Ex. 6, and $\alpha\beta(\gamma+\delta)=\gamma\delta(\alpha+\beta)$, then $a_1a_4^2+2a_3^3-3a_4a_3a_2=0$.

8. Find the equation whose roots are $\alpha+\beta\gamma$, $\beta+\gamma\alpha$, $\gamma+\alpha\beta$, where α, β, γ are the roots of $x^3-ax^2+bx-c=0$.

9. If α, β, γ, δ, be the roots of $x^4-ax^3+bx^2-cx+d=0$, find the equation whose roots are

$$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta,\ \alpha\beta\gamma+\alpha\beta\delta+\beta\gamma\delta, \text{ etc.}$$

10. If α, β, γ, δ, are the roots of $x^4+ax^3+bx^2+cx+d=0$, find the equation whose roots are α^2, β^2, γ^2, δ^2. *(P. U. 1919)*

11. Find the equation whose roots are the squares of the roots of $3x^5-5x^4+x^3-7x^2+2x-1=0$.

12. Find the equation whose roots are the cubes of the roots of $x^3+px^2+qx+r=0$.

Here $y=x^3$. Substitution for x in the given equation gives

$$y+r=-py^{\frac{2}{3}}-qy^{\frac{1}{3}}. \qquad (1)$$

Cubing both the sides of (1) we get

$(y+r)^3=-p^3y^2-q^3y-3pqy(py^{\frac{2}{3}}+qy^{\frac{1}{3}})$. Substitution from (1) for $py^{\frac{2}{3}}+qy^{\frac{1}{3}}$, gives $(y+r)^3=-p^3y^2-q^3y+3pqy(y+r)$.

Transposing and re-arranging the terms, we get

$$y^3+(p^3-3pq+3r)\,y^2+(q^3-3pq+3r^2)y+r^3=0.$$

13. Find the equation whose roots are the cubes of the roots of $2x^3-x^2+2x-3=0$.

14. Find the equation whose roots are the cubes of the roots of $x^4-2x^3-x^2+2x+1=0$.

15. Show that the cubes of the roots of $x^3+ax^2+bx+ab=0$ are given by the equation

$$x^3+a^3x^2+b^3x+a^3b^3=0.$$

16. Show that the roots of $x^3-\frac{1}{3}p^2x-\frac{2}{27}p^3-q^2=0$ differ by a constant quantity from the squares of the corresponding roots of $x^3+px+q=0$.

17. If α, β, γ are the roots of the equation $x^3+px+q=0$, find the equation whose roots are $\frac{1}{\alpha}+\frac{1}{\beta}$, $\frac{1}{\beta}+\frac{1}{\gamma}$, $\frac{1}{\gamma}+\frac{1}{\alpha}$.

18. Solve the equation $x^5-5x^4-5x^3+25x^2+4x-20=0$, whose roots are of the form

$$\alpha,\ \beta,\ -\alpha,\ -\beta,\ \gamma.$$

19. Solve the equation

$$x^5-4x^4-10x^3+40x^2+9x-36=0,$$

given that the roots are of the form

$$\pm a,\ \pm b,\ c.$$

20. α, β, γ be the roots of the equation

$$a_0x^3+3a_1x^2+3a_2x+a_3=0,$$

find the equation whose roots are

$$(\alpha-\beta)(\alpha-\gamma),\ (\beta-\gamma)(\beta-\alpha),\ (\gamma-\alpha)(\gamma-\beta).$$

21. If α, β, γ, δ be the roots of the equation

$$f(x)\equiv a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0,$$

show that if $\alpha\beta=\gamma\delta$, then $a_0a_3^2=a_1^2a_4$. Deduce the equation whose roots are $\frac{\beta\gamma-\alpha\delta}{\beta+\gamma-\alpha-\delta}$, $\frac{\gamma\alpha-\beta\delta}{\gamma+\alpha-\beta-\delta}$, $\frac{\alpha\beta-\gamma\delta}{\alpha+\beta-\gamma-\delta}$.

Solution. Diminish the root of $f(x)$ by z, the required equation is $f(x+z)=0$

i.e., $f(z)+4xf_1(z)+6x^2f_2(z)+4x^3f_3(z)+a_0x^4=0$.....(1)

Denote roots of this equation by α', β', γ', δ', and suppose

$$\alpha'\beta'=\gamma'\delta',$$

i.e.,

$$(\alpha-z)(\beta-z)=(\gamma-z)(\delta-z),$$

then $z=\frac{\alpha\beta-\gamma\delta}{\alpha+\beta-\gamma-\delta}$, which is a desired root.

Now the product of two roots of the equation (1) is equal to the product of the other two, if $a_0 f_1^2(z)=f_3^2(z)f(z)$.

This is the desired equation.

22. If the equation whose roots are the squares of the roots of the cubic $x^3-ax^2+bx-1=0$ is identical with the cubic, prove that either $a=b=0$ or $a=b=3$, or a, b are the roots of $z^2+z+2=0$.

23. If $f(x)$, $f_1(x)$, $f_2(x)$ stand for the expressions

$$a_0x^3+3a_1x^2+3a_2x+a_3,\ a_0x^2+2a_1x+a_2,\ a_0x+a_1$$

respectively, and α, β, γ are the roots of $f(x)=0$, show that

$$a_0 f_1^3(h)=f_2^3(h)f_3(h),$$

if $(\alpha-h)(\beta-h)=(\gamma-h)^2$. Deduce that the equation whose roots are

$$\frac{\beta\gamma-\alpha^2}{\beta+\gamma-2\alpha},\ \frac{\gamma\alpha-\beta^2}{\gamma+\alpha-2\beta},\ \frac{\alpha\beta-\gamma^2}{\alpha+\beta-2\gamma},$$

is

$$af_1^3(x)=f_2^3(x)\,f_3(x).$$

[Hint. Diminish the roots of $f(x)=0$ by h. The new equation has its roots in G. P.]

24. If α, β, γ are the roots of the equation $x^3+px+q=0$, show that the equation whose roots are the six values of $\frac{\alpha}{\beta}$ is

$$q^2(x^2+x+1)^3+p^3x^3(x+1)=0.$$

[Let $\frac{\alpha}{\beta}=z$. Since α is a root of the given equation, $\alpha^3+p\alpha+q=0$, or $\beta^3z^3+p\beta z+q=0\ldots(i)$. Also $\beta^3+p\beta+q=0\ldots(ii)$

Subtraction gives $\beta^2(z^2+z+1)+p=0$.

Again, subtract (i) from z^3 times (ii) ; then eliminate β.]

25. Denoting the roots of the equation $x^4-x+1=0$ by x_1, x_2, x_3, x_4, show that if $y_r=x^3{}_r+x_r$, $(r=1, 2, 3, 4)$, then y_r satisfies the equation $y^4-3y^3+7y^2-7y+5=0$

[*St. Catherine College etc. 1937*]

CHAPTER IV

NATURE OF THE ROOTS AND THEIR LOCATION : SOLUTION OF NUMERICAL EQUATIONS

4·1. The chapter will be concerned with the following points :

1. The number of real and complex roots.
2. The number of positive, negative, and equal roots.
3. The region in which a particular root lies.
4. The region in which all the roots lie.

The information on these points will be useful in the solution of numerical equations to which the last portion of the chapter is devoted.

4·2. Definitions.—When in any polynomial (the terms, of course, being written in order) a + sign follows +, or a − sign follows −, a **continuation**, or **permanence**, of signs is said to occur; but if, on the other hand, a + sign follows −, or a − sign follows +, a **variation**, or a *change* of signs, is said to occur. Thus in the equation $3x^5-x^4-2x^3+3x^2+5x-7=0$, the first continuation of signs occurs at the term $-2x^3$, the second at the term $5x$, while the first variation or change occurs at $-x^4$, the second at $3x^2$, and the third at -7, giving in all two continuations and three variations. The sum of the continuations and the variations is 5, which is the degree of the equation. In general, in a complete equation of the nth degree, there are $n+1$ terms and, consequently, the sum of continuations and changes is always n. The coefficients and their signs bear important relations to the nature of the roots. In the following articles some general theorems will be proved about these relations.

4·3. The Harriot-Descartes' Rule of Signs.

Before we prove this theorem, we take up the following lemma :

If a real polynomial $f(x)$, complete or incomplete, be multiplied by $x-h$, $h>0$, the product will contain at least one more change of sign than the original.

Illustration. Let $f(x)\equiv x^7+x^6+3x^5-x^4-4x^3+6x^2-5x-6$.

We group together consecutive terms with the same signs, and write

$$f(x)\equiv(x^7+x^6+3x^5)-(x^4+4x^3)+6x^2-(5x+6).$$

The number of variations is 3, *i.e.*, one less than the number of groups.

To multiply $f(x)$ by $x-h$, $h>0$, we shall add

$$x f(x)\equiv(x^8+x^7+3x^6)-(x^5+4x^4)+6x^3-(5x^2+6x),$$

and $$-hf(x)\equiv-h(x^7+x^6+3x^5)+h(x^4+4x^3)-6hx^2+h(5x+6).$$

The sign of x^8 in $(x-h)f(x)$ is +ve. The coefft. of x^5 being $-1-3h$, the sign of the first term in the next group is −ve. Similarly, the signs of first terms in subsequent groups are alternately +ve and −ve. There being an addition of one term in the product at the end, the number of variations is *at least* one more than the number in $f(x)$.

That the increase in the number of variations may be more will be seen by giving different values to h. Thus when $h=1$, the variations are increased only by one; but when $h=2$,

$$-hf(x)\equiv-2(x^7+x^6+3x^5)+2(x^4+4x^3)-12x^2+2(5x+6),$$

and $(x-h)\ f(x)$ stands as,

$$(x^8-x^7+x^6)-(7x^5+2x^4)+14x^3-(17x^2-4x)+12.$$

There is an increase of two variations inside the brackets in the first group of terms, and one in the last but one group. Since $4x+12$ is a continuation, the net increase is 3, *i.e.*, 2 more than in the first case when $h=1$.

Proof.

Let the function $f(x)$ be

$$(a_0 x^n+\ldots)-(a_1x^{n_1}+\ldots)+(a_2x^{n_2}+\ldots)-\ldots+(-1)^m(a_mx^{n_m}+\ldots)$$

in which the terms are arranged according to descending powers of x and the terms inside each bracket are positive. The number of variations of signs in $f(x)$ is evidently m.

To multiply $f(x)$ by $x-h$, $h>0$, we have to add the polynomials

$$xf(x)\equiv(a_0x^{n+1}+\ldots)-(a_1x^{n_1+1}+\ldots)+(a_2x^{n_2+1}+\ldots)-\ldots$$
$$(-1)^m(a_mx^{n_m+1}+\ldots).$$

and $$-hf(x)\equiv -h(a_0x^n+\ldots)+h(a_1x^{n_1}+\ldots)-h(a_2x^{n_2}+\ldots)$$
$$+\ldots(-1)^{m+1}h(a_mx^{n_m}+\ldots).$$

Each of these polynomials has m variations, and in the polynomial $xf(x)$, the signs of $f(x)$ are repeated, with degrees of the first terms in brackets being changed to $n+1$, n_1+1, ...

We proceed now to determine the changes of signs in $(x-h)f(x)$. We note that

the signs of terms of degrees $n+1$, n_1+1, n_2+1,...n_m+1 in $(x-h)f(x)$ and $xf(x)$ are the same. This is obvious for the term of degree $n+1$. The term of degree n_1+1 in $(x-h)f(x)$ will be the sum of two terms, *viz.*, $-a_1x^{n_1+1}$ and a term of

the same degree (if it is not zero) from $-h\,f(x)$ which will arise from the first bracket after it has been multiplied by $-h$, and is also therefore negative. Thus the term of degree n_1+1 in $(x-h)\,f(x)$ has the negative sign as in $x\,f(x)$. Similar argument applies to terms of degrees, n_2+1, $n_3+1,\ldots n_m+1$. Thus the variations of signs of $x\,f(x)$, which are m in number, are repeated in $(x-h)\,f(x)$. In addition to this, the last term in $-h\,f(x)$ has the sign opposite to that of x^{n_m+1} in $x\,f(x)$ and, therefore, in $(x-h)\,f(x)$. Thus in the product $(x-h)\,f(x)$, the terms of degrees $n+1$, n_1+1, $n_2+1,\ldots n_m+1$, and the last term have alternately the signs $+$ and $-$, and these variations are $m+1$ in number.

The number of variations in $(x-h)\,f(x)$ may be more than $m+1$. For, it will be noticed that in passing from a $+$ve. to a $-$ve., or from a $-$ve. to a $+$ve. sign (*e.g.*, $+--+--+-$ or $-++--+-+$), an odd number of variations are obtained, and so the gain, if any, is by an even number. Thus if $(x-h)f(x)$ has more than $m+1$ variations, it will have $2k+m+1$, k being a positive integer. These additional variations, of course, will arise from the addition of terms of the same degree in polynomials $x\,f(x)$ and $-hf(x)$, which are put in brackets.

Descartes Theorem. *In an equation $f(x)=0$, the number of positive roots cannot exceed the number of changes of signs of the coefficients, and the number of negative roots cannot exceed the changes of signs of coefficients in $f(-x)$.*

Proof. Let $\phi(x)$ be the product of the factors corresponding to the complex and negative roots, and suppose that a_1, $a_2\ldots\ldots a_p$, $(n>p)$ are the positive roots of $f(x)=0$.

If $\phi(x)$ be multiplied by $(x-a_1)(x-a)\ldots(x-a_p)$ in succession, each multiplication introduces at least one change of sign. This proves the first part.

To prove the second part we notice that the negative roots of $f(x)$ are the positive roots of $f(-x)$, hence follows the second part from the first.

We may note in passing that in a *complete equation* the changes in signs of $f(-x)$ are continuations of $f(x)$. Hence *in a complete equation the number of negative roots cannot exceed the continuations in $f(x)$.*

4·31. Complex roots. Descartes' Rule furnishes useful information about the complex roots of an equation.

Illustration. Let $f(x)\equiv x^7-3x^5+2x^4+x^3-8=0$. There being three changes of sign, the equation cannot have more than three

positive roots. Also $-f(-x)\equiv x^7-3x^5-2x^4+x^3+8=0$, has only two changes of sign; the given equation cannot have, therefore, more than two negative roots. The maximum number of positive and negative roots being 5, and the equation being of the seventh degree, the number of complex roots is at least two.

The above information with regard to complex roots is possible only because the equation is **incomplete.** In a *complete equation, the sum of the number of variations and continuations is exactly equal to the degree of the equation*, and no conclusion can, therefore, be drawn with regard to the existence of complex roots.

4·32. *If an equation has only one change of signs, the equation must have one positive root.*

Examples IV (*a*)

1. Show that the equation $x^3+5x+7=0$ must have one negative and two complex roots.

Obviously, there is no positive root. Also changing $+x$ into $-x$, we see that there is one change of signs. Thus there is one positive root of the new equation, *i.e.*, one negative root of the original equation. The remaining two roots are complex.

2. Find the nature of the roots of the equation

$$x^3+2x-3=0.$$

3. Prove that the equation $3x^5+2x^3-7x^2+8x+9=0$ has *at least* two complex roots.

4. Show that the equation $2x^5+3x^3+9x-7=0$ has no negative root. Hence determine the number of complex and positive roots.

5. If an equation involves no odd power of the variable and all its terms are positive, prove that all the roots are complex.

6. Show that the equation $x^{2n}-1=0$ has only two real roots and the rest are complex. Find the real roots by inspection.

7. Determine the number of complex and positive roots of $x^{2n+1}-1=0$. Also find the real root or roots by inspection.

8. How many complex roots has the equation $x^{2n+1}+1=0$? Find the real root or roots.

9. Apply Descartes' Rule of Signs to the equation

$$3x^4-23x^3+35x^2+31x-30=0,$$

given that all the roots are real.

10 Prove that the equation $x^3+3x+2=0$ has two imaginary roots. (*P. U. 1928*)

11. Show that 4 is the maximum number of complex roots of the equation $x^5-3x^3+5x^2-7x+1=0$.

12. Show that the equation $x^{12}-x^4+x^3-x^2+1=0$ must have at least six imaginary roots. (*P. U. 1931*)

4·33. In order to make further investigations about the rules that govern the nature of the roots and the regions in which they lie, it will be necessary to enquire into the varying values assumed by the polynomial $f(x)$, corresponding to the values taken up by the variable x. Those particular values of x which make $f(x)$ vanish, will constitute obviously the roots of the equation $f(x)=0$, and when the region, in which a particular root lies, is narrowed sufficiently, it will be of great practical help in the actual discovery of the root.

The following theorem points out the direction in which to proceed :—

4·4. Theorem. *If for $x=a$ and $x=b$, the polynomial $f(x)$ has values of $f(a)$ and $f(b)$, of opposite signs, then the equation $f(x)=0$ has at least one real root in the interval (a, b).*

Let $a<b$, $f(a)>0$ and $f(b)<0$.

Now $f(x)=0$ has n roots. Assume that r roots are real and $n-r$ complex. Let $\alpha_1, \alpha_2, \alpha_r$, be the real roots in order of algebraic magnitude.

Then $f(x)\equiv(x-\alpha_1)(x-\alpha_2)\ldots\ldots(x-\alpha_r)\phi(x)$,

where $\phi(x)=0$ gives the complex roots of $f(x)=0$.

For any real value of x, $\phi(x)$ is always positive, for its factors are of the form $(x-\gamma)^2+\delta^2$.

If possible, let no root of $f(x)=0$ lie between a and b ; then for $x=\beta$, where $a\leqslant\beta\leqslant b$,

$$f(\beta)=(\beta-\alpha_1)(\beta-\alpha_2)\ldots\ldots(\beta-\alpha_r)\phi(\beta).$$

Let k roots be $<a$ and $r-k$ roots $>b$.

Now $\phi(\beta)$ and $(\beta-\alpha_1)(\beta-\alpha_2)\ldots\ldots(\beta-\alpha_k)$ are both positive ; and $(\beta-\alpha_{k+1})\ldots\ldots(\beta-\alpha_r)$ is of the same sign as $(-1)^{r-k}$.

$\therefore$ $f(\beta)$ is positive or negative, according as $r-k$ is even or odd.

$\therefore$ $f(x)$ remains of the same sign, positive or negative, throughout the interval, which contradicts the hypothesis.

Hence *at least* one real root of the equation $f(x)=0$ lies between a and b.

4·41. Since $f(x)$ is everywhere continuous, it can, therefore, be completely represented by a continuous curve, a curve without breaks or gaps. A description of the proposition in terms of the graph is, that a continuous curve cannot pass from a point on the positive side to a point on the negative side of the x-axis without cutting that axis *at least* once.

4·42. If $f(a)=A$ and $f(b)=B$, and C is any number between A and B, there must be a value of x between a and b such that $f(x)=C$.

To prove this, put $F(x)=f(x)-C$, and notice that $F(a)=A-C$ and $F(b)=B-C$, of which one is positive and the other negative. Hence there must be a value of x between a and b for which $F(x)=0$ and, consequently, $f(x)=C$.

Ex. 1. Consider the equation $x^3-2x-5=0$.

Since $f(x)$ has only one change of signs, there cannot be more than one positive root; and $\because f(2)$ and $f(3)$ give results with opposite signs, the positive root lies between 2 and 3.

Ex. 2. Let us take another example.

Suppose $f(x)\equiv 25x^3+50x^2-x-2=0$.

Here $f(1)=72$, and $f(-3)=-224$;

hence we conclude that there lies *at least* one real root of the equation between 1 and -3. But $f(x)$ in passing from -224 to 72 may pass through zero more than once. Let us examine it by giving to x some more values between -3 and 1. We see that $f(-1)=24$ and $f(0)=-2$. Thus we conclude :—

(1) $\because f(-3)=-224$, and $f(-1)=24$, } there lies a root between -3 and -1.

(2) $\because f(-1)=24$, and $f(0)=-2$, } there lies a root between -1 and 0.

(3) $\because f(0)=-2$, and $f(1)=72$, } there lies a root between 0 and 1.

So, in the above, $f(x)$ passes from -224 to 72 through zero in the following three stages :—

Firstly : It passes from -224 to $+24$ and thus passes through 0 for a value of x lying between -3 and -1. That value is -2.

Secondly : It passes again from the positive value 24 to the negative value -2 and thus passes through 0 *a second time* for some value of x intermediate between -1 and 0. That value is $-\frac{1}{5}$.

Thirdly : From the negative value -2 it must pass *a third time* through zero to the positive value 72 for some value of x between 0 and 1. That value is $+\frac{1}{5}$.

The above example also shows that when a number of real roots of any equation exist, they may be separated by the help of this Article.

4·43. *The equation $f(x)=0$ has an odd or even (including zero) number of real roots between a and b $(a<b)$, according as $f(a)$ and $f(b)$ are of unlike or like signs.*

Let $\alpha_1, \alpha_2, \ldots \alpha_k$, be the real roots of $f(x)=0$ between a and b,

and $$f(x)=(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_k)\phi(x).$$

Now, $$f(a)=(a-\alpha_1)(a-\alpha_2)\ldots(a-\alpha_k)\phi(a),$$
$$f(b)=(b-\alpha_1)(b-\alpha_2)\ldots(b-\alpha_k)\phi(b).$$

Now $\phi(a)$ and $\phi(b)$ are of like signs, for otherwise a root of $\phi(x)=0$ and, therefore, of $f(x)=0$ other than $\alpha_1, \alpha_2 \ldots\ldots \alpha_k$, will lie between a and b, which is contrary to hypothesis.

Case I. If $f(a)$ and $f(b)$ are of contrary signs, the expressions

$$(a-\alpha_1)(a-\alpha_2)\ldots\ldots(a-\alpha_k), \quad \ldots (i)$$

and $$(b-\alpha_1)(b-\alpha_2)\ldots\ldots(b-\alpha_k) \quad \ldots (ii)$$

are of contrary signs. Also each factor in (i) is negative and each factor in (ii) is positive, hence the number of factors must be odd, *i.e.*, the number of the roots, $\alpha_1, \alpha_2, \ldots\ldots \alpha_k$, must be odd.

Case II. If $f(a)$ and $f(b)$ have the same sign, the expressions, (i) and (ii), have the same sign, and so the number of factors must be even including zero.

In the next Article are given two special cases which are of important application.

4·44. *Every equation of an odd degree must have at least one real root of the sign opposite to that of its last term.*

Let $f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots\ldots+a_{n-1}x+a_n=0$, where a_0, as said before, is always supposed to be positive.

When x becomes infinitely great, (whether positively or negatively) $f(x)$ and a_0x^n are of like signs (1·32).

Hence $f(+\infty)\rightarrow+\infty$, $f(-\infty)\rightarrow-\infty$, since, n is odd. Also $f(0)=a_n$.

Thus if a_n is positive, $f(0)$ and $f(-\infty)$ are of opposite signs and there is at least one real root lying between 0 and $-\infty$, *i.e.*, there is at least one real negative root.

If, on the other hand, a_n is negative, there exists *at least* one real positive root. Thus we establish the proposition.

4·45. *Every equation of an even degree, whose last term is negative, has at least two real roots, one positive and the other negative.*

Here $f(+\infty)\rightarrow+\infty$,

and $f(0)$ is negative, since a_n is negative.

Also $f(-\infty)\rightarrow+\infty$, since n is even.

Hence there is at least one real positive root and one real negative root.

Examples IV (*b*)

1. Show that the equation $3x^3-5x^2+7x-9=0$ must have at least one positive root, and find the interval in which it lies.

2. Show that the equation $x^4-2x^3-13x^2+38x-24=0$ has *one* and *only one* negative root and three positive roots.

3. Given that the real roots of the equation

$$24x^4-100x^3-182x^2+25x+44=0,$$

lie between -2 and 7 ; find also the separate regions in which they lie.

4. Show that the quintic $x^5+x^3-8x-5=0$ cannot have more than three real roots, and prove that it must have three real roots. (*P. U. 1933*)

5. Find the equation whose roots are the squares of the roots of $x^3+4x^2+9x+10-0$.

Hence find the nature of the roots of the given equation.

[The student will find the new equation to be

$$y^3+2y^2+y-100=0,$$

which has only one positive root. Hence the given equation has only one real root, which is obviously negative, and the other two roots are complex.]

6. Find, by the above method, the nature of the roots of the quintic $3x^5-5x^4+x^3-7x^2+2x-1=0$.

7. If a and b be positive, prove that the equation $(x-a)^2(x+b)-x^2=0$ has one negative and two positive roots.

8. If an equation involves no odd power of the variable and all its terms are positive, prove that it can have no real root.

9. Discuss the nature of the roots of the equation $x^3+3Hx+G=0$, by the help of the equation of squared differences.

4·5. Rolle's Theorem.

Between two consecutive roots of $f(x)=0$, lies at least one, and always an odd number of roots of $f'(x)=0$.

Let the consecutive roots a and b, be the multiple roots of order p and q, and let the product of the factors corresponding to the remaining roots be $\phi(x)$, then

$$f(x)=(x-a)^p(x-b)^q\phi(x). \quad \ldots (1)$$

and
$$f'(x)=p(x-a)^{p-1}(x-b)^q\phi(x)+q(x-a)^p(x-b)^{q-1}\phi(x)$$
$$+(x-a)^p(x-b)^q\phi'(x)$$
$$=(x-a)^{p-1}(x-b)^{q-1}[p(x-b)\phi(x)+q(x-a)\phi(x)$$
$$+(x-a)(x-b)\phi'(x)]$$
$$=(x-a)^{p-1}(x-b)^{q-1}\psi(x), \quad \ldots (2)$$

where
$$\psi(x)=p(x-b)\phi(x)+q(x-a)\phi(x)$$
$$+(x-a)(x-b)\phi'(x). \quad \ldots (3)$$

Suppose $b > a$, then since a and b are consecutive roots, $\phi(a)$ and $\phi(b)$ are of the same sign, for otherwise a root of $\phi(x)=0$, and so of $f(x)=0$, will lie between a and b, which contradicts the supposition that a and b are consecutive roots.

Now
$$\psi(a)=p(a-b)\phi(a),$$
$$\psi(b)=q(b-a)\phi(b),$$
which are of contrary signs, and hence an odd number of roots of $\psi(x)=0$, and so of $f'(x)=0$, lies between a and b.

10. *Interpretations of Rolle's Theorem :—*

(1) The reader knows that $f'(x)$ vanishes for those values of x for which the function $f(x)$ has maximum or minimum (turning) values. So the theorem merely asserts, that between two consecutive values of x for which $f(x)$ vanishes, there is an odd number of turning values of the function $f(x)$.

(2) Geometrically, the theorem means that between two consecutive intersections of the curve $y=f(x)$ and the x-axis, $y=0$, there are odd number of points at which the tangents are parallel to the x-axis.

4·51. Conditions for multiple roots.

Equations (1) and (2) show that the root a which occurs p times in $f(x)=0$, occurs $(p-1)$ times in $f'(x)=0$. Similarly, since $f''(x)$ is the derivative of $f'(x)$, it is evident that $f''(x)$ will contain $(x-a)^{p-2}$ as a factor ; $f'''(x)$ will contain $(x-a)^{p-3}$; and so on.

And, conversely, if $f(x)$ and its first $p-1$ derivatives all vanish for $x=a$, then $(x-a)^p$ is a factor of $f(x)$.

For $f(x)=f(a+\overline{x-a})$
$$=f(a)+\frac{(x-a)}{1\,!}f'(a)+\ldots\ldots+\frac{(x-a)^{p-1}}{(p-1)\,!}f^{p-1}(a)$$
$$+\frac{(x-a)^p}{p\,!}f^p(a)+\ldots+\frac{(x-a)^n}{n\,!}f^n(a),$$
and as $f(a)=f'(a)=\ldots\ldots=f^{p-1}(a)=0$;
we get $f(x)=\frac{(x-a)^p}{p\,!}f^p(a)+\ldots+\frac{(x-a)^n}{n\,!}f^n(a)$,
which proves the theorem.

Since multiple roots of orders p, q, r, etc., of $f(x)=0$ are multiple roots of orders $p-1$, $q-1$, $r-1$, etc., of $f'(x)=0$; it, therefore, follows that in order to find such roots we must find the Highest Common Factor of $f(x)$ and $f'(x)$. Let this be $\theta(x)$, then $\theta(x)=0$ will give the multiple roots of $f(x)=0$.

It follows, conversely, that if $f(x)$ and $f'(x)$ have no common factor, the equation $f(x)=0$ has no multiple root.

Also the equation $f(x)=0$ will have a double root, if $f(x)$ and $f'(x)$ vanish simultaneously. Thus the condition of a double root will be the x-eliminant of $f(x)$ and $f'(x)$. Similarly the condition of a triple root will be the x-eliminant of $f(x)=0$, $f'(x)=0$, $f''(x)=0$.

Ex. Solve the equation $x^6-6x^4-4x^3+9x^2+12x+4=0$, given that it has equal roots.

The derivative of the left-hand side is

$6x^5-24x^3-12x^2+18x+12$, *i.e.*, (removing the factor 6),

$$x^5-4x^3-2x^2+3x+2.$$

We now find the H. C. F. to be $x^4+x^3-3x^2-5x-2$, which equated to zero will give the repeated roots.

Now $x^4+x^3-3x^2-5x-2=(x+1)^3(x-2)$.

Hence the given equation has two roots equal to 2 and four roots equal to -1.

4·511. It would be interesting to apply this method to find the condition for equal roots of a cubic or a quartic. The application to a cubic only is shown here. A quartic can be dealt with similarly, though the method is laborious.

Ex. 1. The general cubic $a_0x^3+3a_1x^2+3a_2x+a_3=0$, ...(1) being reduced to the standard form $f(z)\equiv z^3+3Hz+G=0$, ...(2) we need consider only this reduced equation, for when the roots of (2) are equal, the roots of (1) must be equal.

$\because f'(z)=3(z^2+H)$, we find the H. C. F. of z^2+H and the left-hand side of (2):

$$\begin{array}{rl|l} & z^2+H & z^3+3Hz+G \\ & & z^3+Hz \\ \hline (\text{Suppose})\ f_1(z) & \equiv 2Hz+G & 2Hz^2+2H^2 \\ & & 2Hz^2+Gz \\ \hline & & -Gz+2H^2 \\ & & -2GHz+4H^3 \\ & & -2GHz-G^2 \\ \hline (\text{Suppose}) & f_2(z)\equiv & G^2+4H^3 \end{array}$$

In order that the cubic may have equal roots, it is necessary that $G^2+4H^3=0$.

Also the repeated root is given by the equation

$$a_0x+a_1\equiv z=-G/2H.$$

If among the remainders $f_1(z)$, $f_2(z)$, the last alone vanishes, there exist only two equal roots.

The condition for three equal roots will obviously be that each of H and G be zero, separately.

Ex. 2. Find the condition to be fulfilled in order that two roots of the equation $a_0x^3+3a_1x^2+3a_2x+a_3=0$ be equal.

Find also the value of the repeated root.

A double root is a common root of the equations

$$f(x)=a_0x^3+3a_1x^2+3a_2x+a_3=0 \qquad \ldots(i)$$

and
$$\tfrac{1}{3}f'(x)=a_0x^2+2a_1x+a_2=0. \qquad \ldots(ii)$$

Thus to find the condition for equality of two roots, it is necessary to eliminate x between (i) and (ii). This will be done by Bezout's Method which can be given a slightly different presentation from that given in the *Note* to Art. 2·8.

Equation (ii) can be written in the form

$$a_0x^3+2a_1x^2+a_2x=0, \qquad \ldots(iii)$$

$$\therefore \quad a_1x^2+2a_2x+a_3=0. \qquad \ldots(iv)$$

Again,
$$\frac{a_0x^3+3a_1x^2}{a_0x^3+2a_1x^2}=\frac{3a_2x+a_3}{a_2x}, \qquad \text{from } (i) \text{ and } (iii)$$

$$\therefore \quad 2a_0a_2x^2+(a_0a_3+3a_1a_2)x+2a_1a_3=0. \qquad \ldots(v)$$

$\therefore$ from (ii), (iv) and (v), the required condition is

$$\begin{vmatrix} a_0 & 2a_1 & a_2 \\ a_1 & 2a_2 & a_3 \\ 2a_0a_2 & a_0a_3+3a_1a_2 & 2a_1a_3 \end{vmatrix}=0.$$

To find the value of the repeated root, we note that it satisfies the three equations

$$(a_0x+3a_1)x^2+3a_2x+a_3=0,$$
$$a_0x^2+2a_1x+a_2=0,$$
$$a_1x^2+2a_2x+a_3=0.$$

$$\therefore \quad \begin{vmatrix} a_0x+3a_1 & 3a_2 & a_3 \\ a_0 & 2a_1 & a_2 \\ a_1 & 2a_2 & a_3 \end{vmatrix}=0,$$

or
$$2a_0x\begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}+\begin{vmatrix} 3a_1 & 3a_2 & a_3 \\ a_0 & 2a_1 & a_2 \\ a_1 & 2a_2 & a_3 \end{vmatrix}=0.$$

$$\therefore \quad x=-\begin{vmatrix} 3a_1 & 3a_2 & a_3 \\ a_0 & 2a_1 & a_2 \\ a_1 & 2a_2 & a_3 \end{vmatrix} \div 2a_0\begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}.$$

4·52. Between two successive roots of $f'(x)=0$ lies at most one real root of $f(x)=0$.

Let α, β, γ, be three consecutive roots of $f(x)=0$, and let the intervals (α, β), (β, γ) contain the single roots x_1, x_2, respectively, of $f'(x)=0$. If α, β, γ, are in ascending order of magnitude, we have the inequality

$$\alpha<x_1<\beta<x_2<\gamma,$$

which shows that β lies in the interval (x_1, x_2).

Moreover, there cannot lie more than one root of $f(x)=0$ between the two successive roots x_1, x_2, of $f'(x)=0$. For, if there be two, β, β', say, between x_1 and x_2, so that $x_1<\beta<\beta'<x_2$, then by Rolle's Theorem, the interval (β, β') contains a root of $f'(x)=0$, which contradicts the supposition that x_1, x_2, are consecutive roots of $f'(x)=0$.

A root of $f(x)=0$ will lie between the two consecutive roots x_1, x_2, of $f'(x)=0$, only in case $f(x_1)$ and $f(x_2)$ are of different signs. If $f(x_1)$ and $f(x_2)$ are of the same sign, then no root of $f(x)=0$ lie between x_1 and x_2.

4·53. Determination of the number of real roots.

If all the real roots of the equation $f'(x)=0$ could be found, it would be possible to determine the number of real roots of the equation $f(x)=0$.

Let x_1, x_2,.....x_k, be the real roots of the equation $f'(x)=0$, of degree $n-1$, arranged in order, *i.e.*, $x_1<x_2$ .....$<x_k$, and let the series of values

$$-\infty\ ,\ x_1,\ x_2,.....x_k,\ \infty$$

be substituted for x in $f(x)$. When any two of the consecutive values give results of different signs, there is one and only one root of $f(x)=0$ between those values; when they give results of the same sign, there is no root between them. Hence a change of sign, due to the above two successive substitutions in $f(x)$, proves the existence of a real root of $f(x)=0$ between them.

If all the roots of $f(x)=0$ are real, the roots of $f'(x)=0$ will also be all real, by Rolle's Theorem. Similar will be the case for $f''(x)=0$, etc. It follows therefore *$f(x)=0$ has at least as many complex roots as* $f'(x)=0$ has.

Equations like $f'(x)=0$, $f''(x)=0$, etc., which are of one degree less than the one above it [$f(x)=0$, not having all its roots complex], and the roots of which lie severally between consecutive roots of the higher equation, are called *limiting equations*.

Examples IV (*c*)

1. Examine the cubic $4x^3-4x^2-15x+18=0$ for equal roots.

[Here $f(x)=4x^3-4x^2-15x+18$ and $f'(x)=12x^2-8x-15$. The H. C. F. of $f(x)$ and $f'(x)$ is found to be $2x-3$. Hence $\frac{3}{2}$ is a root, which occurs twice in the given cubic. The remaining root is easily found to be -2.]

2. Find if the biquadratic $2x^4-5x^3-21x^2+45x+27=0$ has equal roots. If so, solve it completely.

3. Test the equation $2x^5+x^4-16x^3-8x^2+39x+16=0$ for multiple roots. Solve completely, if multiple roots exist.

4. Show by finding the H.C.F. that the quartic $x^4-4x^3+10x^2-12x+9=0$ has two distinct pairs of equal roots.

Also compute G, H, and I, and show that the general conditions are fulfilled.

5. Show that the quartic $x^4-3x^3-6x^2+28x-24=0$ has three equal roots. Also show that $I=J=0$.

6. Examine by finding the H. C. F., the nature of the roots of the quartic $x^4-12x^3+54x^2-108x+81=0$.

Also compute the values of H, G and I, and show that the general conditions are fulfilled.

7. Solve the equation $x^6-6x^4-4x^3+9x^2+12x+4=0$, given that it has equal roots.

8. Determine the multiple roots of the following equations :

(*i*) $x^5-3x^4-6x^3+10x^2+21x+9=0$, [*Math. Trip.* 1919]

(*ii*) $x^5-x^4-4x^2+7x-3=0$. [*Math. Trip.* 1920]

9. Show that the equation $x^n-1=0$ cannot have equal roots.

10. Show that the equation $1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}=0$ cannot have equal roots.

11. (*i*) Find the range of values of k for which all the roots of the equation $x^4+4x^3-8x^2+k=0$ are real. [*Downing*, 1929]

[*Hint.* If $f(x)=x^4+4x^3-8x^2+k$, $f'(x)=4(x^3+3x^2-4x)$ which vanishes for $x=0, -4, 1$.

Now $f(x)$ is $+ve.$ (when $x\to-\infty$),

$f(x)=k-128$ (when $x=-4$), $f(x)=k$ (when $x=0$),

$f(x)=k-3$ (when $x=1$), $f(x)$ is $+ve.$ (when $x\to\infty$).

Hence in order that all the roots may be real, $k-128$ should be $-ve.$, $k+ve.$, $k-3$ $-ve.$; and these conditions are satisfied if $0<k<3$.]

Here $x^3+3x^2-4x=0$ is the limiting equation of the proposed equation.

(*ii*) Find the value of k for which $x^3+4x^2+5x+2+k=0$ has equal roots. Also find those roots. [*P. U. 1921*]

Find the range of values of k for which the above equation has real roots.

12. Discuss the nature of the roots of the equation $x^3+3Hx+G=0$, by the methods of the present Chapter.

13. Show that the equation $x^5-5px^3+2q=0$ will have three real roots, provided $27p^5>q^2$. [*Math. Tripos I, 1918*]

14. Discuss the nature of the roots of the equation

$$x^3-3x^2-72x+9=0,$$

by finding the maxima-minima of the given equation. (*Queen's*)

15. Determine for what range of numerical values of k, the roots of $2x^3-9x^2+12x-k=0$ are all real. [*Math. Trip.*]

16. Find the values of a for which the equation $ax^3-9x^2+12x-5=0$ has equal roots, and solve the equation in one case. [*Math. Trip. 1913*]

[**Hint** : Using $G^2+4H^3=0$, a is found to be 2, or $\frac{54}{25}$; Or, First change the equation into its reciprocal.]

17. Determine the value of a such that the equation $x^3-12x+a=0$ has only one real root. [*Math. Trip. 1923*]

18. Show that $x^4-32x+7=0$ has two positive roots, one greater than 2, and the other less than 2 ; and the remaining two roots are complex.

19. Find the range of values of k for which the equation

$$x^4-14x^2+24x-k=0,$$

has all its roots real. [*Math. Trip. 1915 ; Downing, 1932*]

20. Find the limits between which a lies so that the equation $3x^4-4x^3-12x^2+a=0$ has all its roots real. (*Ecole Polytechnique*)

21. The equation $x^n-qx^{n-m}+r=0$ has two equal roots, if

$$\left\{\frac{q}{n}(n-m)\right\}^n=\left\{\frac{r}{m}(n-m)\right\}^m.$$

[**Solution** : The double root is also a root of

$f'(x)\equiv nx^{n-1}-(n-m)qx^{n-m-1}=0$, or $nx^m=(n-m)q$...(i)

and also of $xf'(x)-nf(x)\equiv mqx^{n-m}-n^r=0$, or $x^n=\frac{nr}{mq}x^m$ ...(ii)

Eliminating x betwcen (i) and (ii), we get the desired result.]

22. Show that the equation $(x-p)^3+(x-q)^3+(x-r)^3=0$ has one real and two complex roots.

23. If the equations $x^3+2x^2+3x+6=0$ and $x^2+ax-6=0$ have a common root, find the possible values of a, and the corresponding common roots.

4·6. Limits of the roots. As already hinted, it is necessary to know the region, in which the real roots of an equation

lie, before we can proceed to find their actual or approximate values. As regards the roots whose approximate values only can be found, we shall further require the region in which each root lies separately. As a first step, we shall find the region in which *all* the real roots lie and shall try to narrow it as much as may conveniently be done.

Definitions. A positive number, below which all the positive roots lie, is called an **Upper Positive Limit**, or shortly, an **Upper Limit** (of the positive roots), or a **Superior Limit.**

A number, above which all the positive roots lie, is called a **Lower Positive Limit**, or shortly, a **Lower** or **an Inferior Limit** (of the positive roots).

Similarly, a negative number, below which (numerically) all the negative roots lie, is called a Superior Limit of the negative roots, and a negative number, above which (numerically) all the negative roots lie, is called an Inferior Limit of the negative roots.

We shall first establish propositions for finding Superior Limits of positive roots, and shall show afterwards, that other limits can easily be found, when we know these superior limits.

A superior limit of the positive roots and a superior limit of the negative roots will obviously give the region in which all the real roots lie.

4·61.—Upper Limit. Here is a method, for finding the upper limit, *which* consists simply in *grouping* the terms together in such a way that the smallest value of x, or any value greater than that smallest value, will make each group of terms positive. Before giving the general proposition, we shall explain it by taking a few examples.

Ex. 1. Let $f(x) \equiv x^5+3x^4-8x^3-20x^2+x-2=0$.

The terms in $f(x)$ can be grouped as $x^3(x^2-8)+3x^2(x^2-\frac{20}{3})+(x-2)$, which shows that if x has the value 3, or any value > 3, each of the groups, and hence $f(x)$, will be +ve. Hence 3 is a superior limit.

Ex. 2. $3x^5-2x^4-8x^3-26x^2+x-2=0$.

Here the highest term, $3x^5$, can be split up so that we can take a part of it with each of the negative terms, which follow it. Thus we have $x^4(x-2)+x^3(x^2-8)+x^2(x^3-26)+(x-2)$.

Hence 3 is a superior limit.

Ex. 3. $x^6-13x^5-14x^4+x^2-3x-80=0$.

The polynomial may be written as

$$x^4(x^2-13x-14)+x^2-3x-80.$$

Now the first group, $x^2-13x-14=(x+1)(x-14)\geqslant 0$ for every value of $x\geqslant 14$. Also 14, or any value >14, makes the second group, $x^2-3x-80$, $+ve$. Hence 14 is a superior limit.

Ex. 4. $f(x)\equiv x^4-9x^3+25x^2-x+5=0$.

Here $f(x)$ may be put as $x^2(x^2-9x+22)+3x(x-\frac{1}{3})+5$.

The group, $x^2-9x+22$, or $(x-\frac{9}{2})^2+\frac{7}{4}$ is positive for *all* real values of x, and the remaining terms give a positive result for every value of $x>\frac{1}{3}$, or equal to $\frac{1}{3}$. Hence $\frac{1}{3}$ is a superior limit of the positive roots.

It is a useful artifice, in certain cases, to make a quadratic group of terms like $x^2-9x+22$, which, when equated to zero, will give either complex or coincident roots. The case when the quadratic group has real roots has been illustrated in Ex. 3 above.

The principle, underlying the above illustrations, is established in the following proposition.

4·62. Method of Grouping :—

We put the equation in the form

$$f(x)\equiv\phi(x)-\phi_1(x)+\phi_2(x)=0,$$

where $\phi(x)$ denotes the sum of all the terms preceding the first negative term ; $-\phi_1(x)$, the sum of all negative terms ; and $\phi_2(x)$, the sum of all other positive terms. Then the least number k, which renders the difference, $\phi(x)-\phi_1(x)$, positive, is the superior limit of the positive roots. For, if x^m be the lowest term in $\phi(x)$, we can write the difference in the form

$$x^m\left\{\frac{\phi(x)}{x^m}-\frac{\phi_1(x)}{x^m}\right\},$$

where $\dfrac{\phi(x)}{x^m}$ consists of terms, with positive (including zero) powers of x ; and $\dfrac{\phi_1(x)}{x^m}$, with negative powers of x only. It is clear that any value of x greater than k, will render the difference positive, and so $f(x)$ also positive, and no value of x greater than k can make $f(x)$ vanish ; hence k is the superior limit.

The method can be generalized so as to include cases, when $f(x)$ is decomposed in several groups of the form $\phi(x)-\phi_1(x)$, where the coefficients of $\phi(x)$ and of $\phi_1(x)$ are positive, and each term of $\phi_1(x)$ is of a degree less than those of the terms of $\phi(x)$. Then the least value of x, which renders each group positive, is the superior limit of positive roots.

4·63. Inferior Limits of positive roots.—Transform $f(x)=0$ into a reciprocal equation, by putting $x=\frac{1}{y}$. Now if p is found to be a superior limit of the new equation in y, $\frac{1}{p}$ will be the inferior limit of the positive roots of $f(x)=0$.

4·64. Limits of Negative Roots.—Transform $f(x)=0$ into another equation by putting $x=-y$. The negative roots of $f(x)=0$, will become the positive roots of the new equation in y, and if p and q are found to be the superior and inferior limits of the positive roots of this equation, $-p$ and $-q$ will be the superior and inferior limits of the negative roots of $f(x)=0$.

4·65. Newton's Method for finding the upper limit.

If $f(x)$ and all its derivatives are positive for $x=h$, then h is an upper limit of the positive roots.

Proof : Diminish the roots of $f(x)$ by h, by the transformation

$$x-h=y, \quad \text{or} \quad x=y+h.$$

$$\therefore \quad 0=f(x)=f(h+y)$$

$$=f(h)+yf'(h)+\frac{y^2}{2!}f''(h)+\ldots+\frac{y^n}{n!}f^n(h).$$

Since $f(h)$, $f'(h),\ldots f^{n-1}h$ are all positive, therefore, for all values of $y\geqslant 0$, *i.e.*, $x>h$, $f(h+y)$ remains positive, and therefore cannot vanish. Thus for any value of $x>h$, $f(x)$ cannot vanish, and therefore no root of $f(x)=0$ exceeds h.

Examples IV (*d*)

1. Find the superior and inferior limits of the positive and negative roots of the equation $x^4-8x^3-12x^2+60x+63=0$.

Writing it as $x^2[(x-4)^2-28]+60x+63$, it is seen that the superior limit of positive roots is 10. Putting $x=\frac{1}{y}$, we get $63y^4+60y^3-12y^2-8y+1=0$, which gives $\frac{1}{2}$ as the superior limit of its positive roots. Hence 2 is the inferior limit of the positive roots of the given equation.

Again, putting $x=-y$, we get $y^4+8y^3-12y^2-60y+63=0$, which gives, by grouping the terms, 4 as a superior limit.

The inferior limit is found to be $\frac{1}{2}$. Hence the limits of negative roots are -4 and $-\frac{1}{2}$. These limits may be compared with the roots of this equation, which have been found in Ex. 3. Examples I(*c*) to be 8·5, 3, -1, $-2·5$.

[It will be noticed that the inferior limits found are generally very near zero and often useless. On this account, positive

superior and negative superior limits are considered quite sufficient for all practical purposes.]

2. Find the superior and inferior limits of the positive and negative roots of the equation $x^4+2x^3-25x^2-26x+120=0$.

3. Find the limits between which the real roots of the equation $x^4+x^3-2x^2+4x-24=0$ may lie.

The superior limit of positive roots is found by grouping the terms as $(x^4-24)+x^2(x-2)+4x$, to be 3.

Again, here $f(-x)\equiv x^4-x^3-2x^2-4x-24=0$.

Since the first positive term in $f(-x)$ is followed by 4 negative terms, it will be useful here to multiply throughout by 4, and then group the terms as $(x^4-4x^3)+(x^4-8x^2)+(x^4-16x)+(x^4-96)$, or, as $x^3(x-4)+x^2(x^2-8)+x(x^3-16)+x^4-96$, which shows that 4 is a superior limit of the positive roots of $f(-x)=0$. Thus the required limits are 3 and -4.

4. Show that the real roots of $x^4-44x^2+112x-384=0$, lie between 7 and -12.

5. Show that -5 is a negative superior limit and 0 is positive limit of the roots of

$$x^4+10x^3+35x^2+50x+24=0.$$

[Take a quadratic group of terms.]

6. Find limits for the roots of

$$16x^5+80x^4-96x^3+4x^2+23x-6=0.$$

7. Find the superior limit of the positive roots of

$$2x^7-3x^6+5x^5+7x^4-18x^3+10x^2+8x-20=0.$$

8. Find the superior limit of the positive roots of

$$x^5+3x^4+x^3-8x^2-51x+18=0.$$

9. Find the superior limit of the roots of

$$3x^5-2x^4-8x^3-26x^2+x-2=0.$$

10. Find the superior limit of the positive roots of

$$x^8-x^7+5x^6-15x^5-47x^4+x^3-711x^2-313x+1=0.$$

(Peterson)

$$f(x)=x^8-x^7+5x^6-15x^5-47x^4+x^3-711x^2-313x+1=0,$$
$$f'(x)=8x^7-7x^6+30x^5-75x^4-188x^3+3x^2-1422x-313,$$
$$f''(x)=56x^6-42x^5+150x^4-300x^3-564x^2+6x-1422,$$
$$f'''(x)=336x^5-210x^4+600x^3-900x^2-1128x+6$$
$$=6(56x^5-35x^4+100x^3-150x^2-188x+1),$$
$$f^{iv}(x)=24(70x^4-35x^3+75x^2-75x-47),$$
$$f^{v}(x)=120(56x^3-21x^2+30x-15),$$
$$f^{vi}(x)=360x(56x^2-14x+10),$$
$$f^{vii}(x)=5040(8x-1).$$

Now $f^{vii}(1)>0$. Also for $x=1$, $f^{vi}(x)>0$, $f^{v}(x)>0$, but $f^{iv}(1)<0$. So we try $x=2$. For this value of x, $f^{vii}(x)$, $f^{vi}(x)$,

$f^{v}(x)$ will necessarily be positive. It is seen that for $x=2$, $f^{iv}(x)>0$, $f'''(2)$ is zero, but $f''(x)<0$ and $f''(3)>0$. $f'(3)>0$, but $f(3)<0$ and $f(4)>0$. Thus 4 is the superior limit of the positive roots.

4·63. Location of the Roots. When we have found limits between which the positive and the negative roots lie, we shall find it necessary, in most cases, to know the region in which each real root lies separately. That object may be achieved in simpler cases with the help of the methods already explained; while for certain other equations, it will be necessary to use one of the theorems enunciated by Budan, Fourier, or Sturm. Without going into the exposition of any of these theorems, we shall simply *assume* the statement of Budan's Theorem and apply it in a few cases, where it is deemed necessary.

To start with, we shall show by a few examples how roots may generally be separated unless they lie very close to each other.

Ex. 1. Let us consider the equation

$$f(x)\equiv 4x^4-24x^3+31x^2+6x-8=0.$$

The real roots lie between 6 and -1.

Since $f(-x)\equiv 4x^4+24x^3+31x^2-6x-8=0$,

there exists only one negative root, and that lies obviously between 0 and -1. So either there are three real roots between 0 and 6, or only one.

Again, $f(0)=-8$. $f(1)=+9$. $f(3)=-35$, $f(5)=297$.

Hence, one positive root lies between 0 and 1, another beween 1 and 3, and the third between 3 and 5. In order to find closer limits we shall find $f(2)$ and $f(4)$. Here it happens that $f(2)=0$ and $f(4)=0$, giving 2 and 4 as two of the roots.

Ex. 2. Separate the roots of the equation

$$x^5-4x-2=0. \qquad [P.\ U.\ 1917]$$

The real roots lie between -2 and $+2$. Obviously, there is one and only one positive root. Since here

$$f(-x)\equiv x^5-4x+2=0,$$

there cannot be *more* than two negative roots.

Again, $f(-2)=-26$, $f(-1)=1$, $f(0)=-2$, $f(1)=-5$, $f(2)=22$.

Hence one root lies berween -1 and -2, another between 0 and -1, and the third between 1 and 2.

4·631. Budan's Theorem. When two roots are very close to each other, the best way to separate them will lie in the course of the actual process, and will be explained later, when we shall actually find such roots. The theorem of Budan, which will be used there, is enunciated below :—

Let a and b be real numbers, $a<b$, neither being a root of $f(x)=0$, an equation of degree n with real coefficients. Let V_a denote the number of variations of the signs of

$$f(x), f'(x), \ldots f^{(n)}(x),$$

for $x=a$, after the vanishing terms have been deleted. Then $V_a - V_b$ is either the number of real roots of $f(x)=0$ between a and b, or exceeds the number of those roots by an even integer. A root of multiplicity m is counted as m roots.

Illustration. Let $f(x)=x^5-4x-2$. Take $a=1$, $b=2$.

	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{iv}(x)$	$f^{v}(x)$	
	x^5-4x-2	$5x^4-4$	$20x^3$	$60x^2$	$120x$	120	
Signs when $x=1$ are	−	+	+	+	+	+	$\therefore V_1=1$.
Signs when $x=2$ are	+	+	+	+	+	+	$\therefore V_2=0$.

Hence there is exactly one real root between 1 and 2.

Signs when $x=0$ are	−	−	0	0	0	+	$\therefore V_0=1$.

Hence there is no root between 0 and 1.

We may note that $f(a)$, $f'(a)$,..... $f^n(a)$ are precisely the coefficients which appear in the reduced equation when its roots are diminished by a. Consequently, the rule in the following form is often convenient :—

If the roots of any equation be diminished first by a and then by b, of which a is the lesser, then the number of real roots between a and b cannot be greater than the excess of the number of changes of signs in the first transformed equation over the number in the second.

Since the superior limit of the positive roots is 2 ; when the roots are diminished by 2, there will not remain any positive root in the transformed equation and, consequently, there will be no change of sign (Descartes' Rule) ; so we could have anticipated that all the signs would be positive.

It will be interesting to note that Descartes' Rule of Signs is included in Budan's Theorem as a special case. For, if we substitute $x=0$ in $f(x)$ and its derivatives, we get precisely the coefficients of $f(x)$ in order, so that V_0=the variations in $f(x)$. For $x\to+\infty$, those functions have the same sign as a_0, *i.e.*, $+ve$. Hence $V_0 - V_\infty = V_0$, *i.e.*, variations in $f(x)$.

Examples IV (*e*)

1. Locate the roots of the equation $x^3-3x^2-2x+5=0$.
2. Investigate the nature and position of the roots of

$$f(x)\equiv x^4-x^3-x^2-8x+9=0. \qquad [P.\ U.\ 1928]$$

Here $f(-x)\equiv x^4+x^3-x^2+8x+9=0$. Since $f(-x)$ may be written as $x^4+x(x^2-x+8)+9=0$, and $x^2-x+8=0$ has complex

roots, $f(-x)$ is positive for *all* positive values of x. Hence $f(-x)=0$ has no positive root and, consequently, $f(x)=0$ has no negative root.

3. Locate the roots of the equation

$$f(x)\equiv 81x^5-207x^4-9x^3+89x^2+2x-8=0.$$

Roots lie between 4 and -1.

Maximum number of positive roots is 3 and of negative roots is 2.

Now, $f(-1)$ is $-ve.$; $f(0)=-ve.$; $f(1)$ is $-ve.$; $f(2)$ is $-ve.$; $f(3)$ is $+ve.$; $f(4)$ is $+ve.$

By Descartes' Rule, we only get that one root, or three roots, lie between 2 and 3. So we apply Budan's Theorem, and when roots are diminished by -1, 0, 1, 2, 3, respectively, we shall get the following scheme of signs :—

(-1)	$+$	$-$	$+$	$-$	$+$	$-$
(0)	$+$	$-$	$-$	$+$	$+$	$-$
$(+1)$	$+$	$+$	$-$	$-$	$-$	$-$
(2)	$+$	$+$	$+$	$+$	$+$	$-$
(3)	$+$	$+$	$+$	$+$	$+$	$+$

From the above, we conclude that (*i*) all the roots lie between -1 and 3; since all the signs are positive when roots are diminished by 3, there is no root above 3, (*ii*) there is no root between 1 and 2, (*iii*) there is one between 2 and 3, (*iv*) not more than two between 0 and 1, and (*v*) not more than two between 0 and -1.

Thus there is one root between 2 and 3; and if negative roots exist, they are two in number and lie between 0 and -1; similarly, if there are three positive roots, the other two lie between 0 and 1. To ascertain the existence of real roots for the intervals between -1 and 0, and 0 and 1, a closer examination will be necessary, and when we further use this method we shall have actually to separate the nearly equal roots, if they exist. But if they do not, this method will fail to give us any *positive* result.

4. Investigate the nature and position of the roots of the equation $4x^3-10x^2+3x+4=0$.

There is only one negative root. For the other roots, we see $\frac{5}{2}$ is a superior limit of the positive roots, if they exist.

Diminishing the roots by 0, 1, and 2, we have the following sequence of signs :—

(0)	$+$	$-$	$+$	$+$
(1)	$+$	$+$	$-$	$+$
(2)	$+$	$+$	$+$	$+$

Hence the two positive roots, if they exist, lie between 1 and 2. The positive roots exist if all the roots are real, *i.e.*, if G^2+4H^3 is negative. Here $G=\frac{808}{27}$ and $H=-\frac{64}{9}$;

hence $G^2+4H^3=\frac{64}{27^2}(101^2-128^2)$, which is $-ve$.

5. Locate the roots of the equation $x^3-4x^2-3x+18=0$.

6. Show that $x^3+3x^2+x+2=0$ has only one real root, and locate it between two consecutive integers. (*St. Catherine, 1931*)

4·7. Solution of Numerical Equations. We are now in a position to take up the actual problem of finding the real roots of Numerical Equations.

The real roots may be divided into two classes; *Commensurable* or rational, and *Incommensurable* or irrational. Roots whose values are given by integers, rational fractions, or recurring or non-recurring decimal fractions, which are always reducible to rational fractions, are called *commensurable*; while other real roots whose values cannot be so expressed are called *incommensurable*.

Thus an incommensurable root may be expressed by an irrational number, or by a decimal fraction which does not terminate, and is at the same time non-recurring. It will be as a decimal fraction of this form that we shall find our incommensurable roots and though the value of a root in this case cannot be *exactly* found, yet our methods will enable us to find the root, correct to as many decimal places as we like.

Methods of approximating to real roots of Numerical Equations have been given by Newton, Lagrange, Graffe, Horner, and others, but the method given by Horner approximating to both rational and irrational roots, is the best and we shall confine ourselves to this method of approximation.

For rational roots, however, there is the obvious method of ordinary division, but Newton's Method of Divisors is generally more suitable and will be explained here.

4·71. *If an equation with integral coefficients and the coefficient of its highest term unity, has a commensurable root, it must be a whole number.*

For, if possible, let p/q, reduced to its lowest terms, be a root of the equation $x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots\ldots+a_{n-1}x+a_n=0$, where a's are all integers. Substituting p/q for x in the equation we have

$$\frac{p^n}{q^n}+a_1\frac{p^{n-1}}{q^{n-1}}+a_2\frac{p^{n-2}}{q^{n-2}}+\ldots\ldots+a_{n-1}\frac{p}{q}+a_n=0.$$

Multiplying throughout by q^{n-1}, and transposing the fractional term, we find

$$-\frac{p^n}{q}=a_1p^{n-1}+a_2p^{n-2}q+\ldots\ldots+a_{n-1}pq^{n-2}+a_nq^{n-1}.$$

Now the right-hand side is integral; and p/q is fractional, since, by hypothesis, p and q have no common factor. But it is impossible for a fraction to be equal to an integer. Hence the equation cannot have a fractional root. and all the commensurable roots must be integers.

25. *For an equation, all of whose co-efficients are integers, the coefficient of the highest power of the variable being unity, an integral root is a divisor of the constant term.*

Let h be a root of the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_{n-1}x+p_n=0,$$

where all the p's are integers, then

$$h^n+p_1h^{n-1}+\ldots\ldots+p_{n-1}h+p_n=0.$$

Since h divides every term up to the last but one, it must divide the last term also, *i.e.*, p_n.

Hence, *in order to determine the integral roots, each divisor of* p_n *may be tested.* The most obvious method is by the *synthetic division.* But this method has been superseded by Newton's Method of Divisors explained in the next Article.

It is evident that any divisor that does not lie between the superior and the inferior limits need not be tested.

Note.—If the coefficient of the highest term is not unity in a given equation, the equation can easily be transformed into another with that coefficient unity and all other coefficients integral—[see § 3·22]. Thus all *commensurable* roots are reduced to *integral* roots, and in this way the commensurable roots of *any* equation can all be found.

4·8. Newton's Method of Divisors.

Let us take the general equation

$$f(x)\equiv a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n=0,$$

where a's are all integers.

Suppose that h is an integral root of this equation and that when $f(x)$ is divided by $x-h$, the quotient, without any remainder, is $b_0x^{n-1}+b_1x^{n-2}+b_2x^{n-3}+\ldots\ldots+b_{n-2}x+b_{n-1}$,

so that $a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots\ldots+a_{n-1}x+a_n$

$$\equiv(b_0x^{n-1}+b_1x^{n-2}+b_2x^{n-3}+\ldots\ldots+b_{n-2}x+b_{n-1})\,(x-h).$$

Equating the coefficients of the corresponding powers of x in the identity, we get the relations.

$$\left.\begin{aligned} &a_0=b_0,\ a_1=b_1-hb_0,\ a_2=b_2-hb_1,\ldots\ldots a_{n-2}=b_{n-2}-hb_{n-3},\\ &a_{n-1}=b_{n-1}-hb_{n-2},\ a_n=-hb_{n-1}. \end{aligned}\right\} \quad (1)$$

Since h and all the a's are integers, all the b's also must be integers.

Now from relations (1), taking them in the reverse order, we get

$$\left.\begin{aligned} &\frac{a_n}{h}=-b_{n-1},\ \frac{-b_{n-1}+a_{n-1}}{h}=-b_{n-2},\\ &\frac{-b_{n-2}+a_{n-2}}{a}=-b_{n-3},\ldots\ \frac{-b_2+a_2}{h}=-b_1,\\ &\frac{-b_1+a_1}{h}=-b_0,\ -b_0+a_0=0. \end{aligned}\right\} \quad \ldots(2)$$

Relations (2) suggest the following method for testing any divisor h of a_n for a root :—

First divide a_n by h, and add the quotient $-b_{n-1}$ to a_{n-1}; divide the sum again by h, and add the quotient $-b_{n-2}$ to a_{n-2}; continue this process of division n times, until we get $-b_0$ as the result of division of $-b_1+a_1$ by h: lastly, $-b_0+a_0=0$. If at any stage of division, we get a fractional quotient, that would show that h is not a root, and the process must stop. If, on the other hand, all quotients are integers, and the last quotient added to a_0, gives zero, h is shown to be a root.

When h is a root, the coefficients of the quotient

$$b_0x^{n-1}+b_1x^{n-2}+b_2x^{n-3}+\ldots\ldots+b_{n-2}x+b_{n-1} \quad \ldots(3)$$

are found in the reverse order by changing the signs of the successive quotients, $-b_{n-1}$, $-b_{n-2},\ldots\ldots-b_0$, in the above process.

When one divisor of a_n has been tested to be a root, the next divisor may more conveniently be tested on the depressed equation, obtained by equating (3) to zero, and so on.

The process may be exhibited as below to attain facility in working :—

a_n	a_{n-1}	$a_{n-2}\ldots\ldots$	a_2	a_1	a_0
	$-b_{n-1}$	$-b_{n-2}\ldots\ldots$	$-b_2$	$-b_1$	$-b_0$
	$-hb_{n-2}$	$-hb_{n-3}\ldots.$	$-hb_1$	$-hb_0$	0

Explanation.—The first line shows the coefficients of the given equation written from the end in order, the second line shows the successive coefficients in the reverse order (with signs changed) of the quotient and the third gives the results of adding the coefficients of the quotient to the corresponding coefficients of the given equation; these successive results of

summation serve, in turn, as numbers to be divided by the divisor h. If h is a root, the numbers in the second line must successively be obtained to be integers; the process must, therefore, cease as soon as we get a fractional quotient, showing that h cannot be a root; also the last number in the third line must be zero, when h is a root.

The method has been explained for the general case when the coefficient of x^n is any integral number a_0, but it is obvious that from such an equation only *integral* roots will be obtained; so in order to obtain all the *commensurable* roots including the fractional roots, the given equation must be reduced to one with coefficient of x^n unity, before using the Method of Divisors.

We shall illustrate the method by taking the equation

$$x^4+x^3-2x^2+4x-24=0. \qquad (P.\ U.\ 1929)$$

The divisors to be tested are ± 3, ± 2. since the limits are 3, -4.

We test 3 :

-24	4	-2	1	1
	-8			
	-4			

So we reject 3 at the very first step.

Now we test 2 :

-24	4	-2	1	1
	-12	-4	-3	-1
	-8	-6	-2	0

Thus 2 is a root and the reduced equation is

$$x^3+3x^2+4x+12=0.$$

The student may easily test the remaining divisors, -2 and -3, on this reduced equation, and he will find -3 to be another root.

It is obvious that the work will generally be much less in this method than in that of Synthetic Division.

Examples IV (f)

1. Find the commensurable roots of

$$x^4-8x^3-12x^2+60x+63=0.$$

As seen in Ex. 1, Examples IV (d), the roots lie between -4 and 10.

Hence the divisors to be tested are 9, 7, 3, 1, -1, -3.

2. Solve completely the equation $2x^3-31x^2+112x+64=0$,

In order to obtain *all* the commensurable roots we transform the given equation to $y^3-31y^2+224y+256=0$, so that $y=2x$.

L. H. S.$=y\left[\left(y-\frac{31}{2}\right)^2-\frac{65}{4}\right]+256$. Now 20 or any number greater than 20 makes the expression within brackets positive, and so the whole expression is $+ve$. for $y=20$; hence 20 is the superior limit of the positive roots.

Hence the roots of the new equation lie between 20 and -2 Hence the divisors to be tested are 16, 8, 4, 2, 1, -1.

We commence with 16 :—

256	224	−31	1
	16	15	−1
	240	−16	0

$\therefore$ 16 is a root, and the depressed equation is

$$y^2-15y-16=0,$$

whose roots are found, in the ordinary way, (being a quadratic) to be 16 and -1.

So the roots of the given equation are 8, 8, $-\frac{1}{2}$.

3. Solve the equation

$$32x^5-360x^3+540x^2-243=0. \quad [P.\ U.\ 1923)$$

The transformed equation with the coefficient of x^5 unity, is $x^5-45x^3+135x^2-243=0$. The limits of roots for the new equation are 7 and -10. Hence the divisors to be tested are

3, 1, -1, -3, -9.

We start with 3 :

−243	0	135	−45	0	1
	−81	−27	36	−3	−1
	−81	108	−9	−3	0

$\therefore$ 3 is a root, and the depressed equation is

$$x^4+3x^3-36x^2+27x+81=0.$$

1 and -1 are obviously not the roots.

So we next test -3.

81	27	−36	3	1
	−27			
	0			

$\therefore$ -3 is not a root, and we next test -9 :

81	27	−36	3	1
	−9	− 2		
	18	−38		

$\therefore$ -9 is also not a root. So we must see if the commensurable root 3 is repeated :

81	27	−36	3	1
	27	18	−6	−1
	54	−18	−3	0

So 3 is repeated, and we try if it occurs once again :—

$$\begin{array}{rrrr} -27 & -18 & 6 & 1 \\ & -9 & -9 & -1 \\ \hline & -27 & -3 & 0 \end{array}$$

It should be noted that if an integer root h occurs thrice, as here, h^3 must be a factor of the last coefficient, h^2 of last but one, h of last but two : and similarly for other cases.

The finally depressed equation is the quadratic $x^2+9x+9=0$, whose roots are $\frac{-9\pm3\sqrt{5}}{2}$.

Hence the required roots are $\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{-9\pm3\sqrt{5}}{4}$.

4. Find the commensurable roots of $x^4+3x^3+2x^2-3x-3=0$.

5. Obtain the integer roots of

$$x^4+2x^3-13x^2-38x-24=0. \qquad [P.\ U.\ 1932]$$

6. The roots of the equation $x^4-x^3-30x^2-76x-56=0$ lie between -6 and 12, find them. (*P. U. 1939*)

4·81. When the absolute term a_n of $f(x)$ is a number possessing many divisors within the limits of real roots, Newton's Method also becomes laborious if every such divisor is to be tested. But by using the method to be explained here, the number of such divisors can be reduced appreciably.

When h is an integer root, $f(x)$ being divisible by $x-h$, the coefficients in the quotient are all integers: so that if x be given any integral value m, $f(m)$ will be divisible by $m-h$, giving an integral quotient.

For the sake of convenience, we may put $x=1$ and -1, and say that $f(1)$ and $f(-1)$ are exactly divisible by $1-h$ and $-1-h$, or changing the signs, by $h-1$ and $h+1$, respectively. So we get the following working rule :—

Calculate $f(1)$ and $f(-1)$; such of the divisors as decreased by 1 fail to divide $f(1)$, are to be rejected; and similarly the divisors, which increased by 1 fail to divide $f(-1)$, are to be rejected. If, however, $f(1)$ vanishes, 1 is seen to be a root; or if $f(-1)$ vanishes, -1 is seen to be a root. When 1, or -1 is a root, the polynomial can be reduced by division by the corresponding factor $x-1$, or $x+1$. Suppose $f(x)=(x-1)\,\phi(x)$; in this case calculate $\phi(1)$, and proceed as before with the process on $\phi(1)$. But if $\phi(1)$ is also zero, let $f(x)=(x-1)^2\theta(x)$; now proceed with the process on $\theta(x)$, and so on. Similar is to be the procedure when $f(-1)$ vanishes.

Examples IV (*g*)

1. Solve the equation $x^4-44x^2+112x-384=0$, given that it has got two commensurable roots.

The real roots lie between 7 and -12, and the possible divisors are 6, 4, 3, 2, 1, -1, -2, -3, -4, -6, -8.

Here $f(1)=-315$ and $f(-1)=-539$.

First of all, 1 and -1 are not the roots and are to be struck off. Again, 3, -3 decreased by 1, fail to divide $f(1)$, *i.e.*, -315, and are to be rejected.

Next 4, -6 increased by 1, fail to divide $f(-1)$, *i.e.*, -539, and thus out of 6, 4, 2, -2, -4, -6, -8; 4 and -6 being further struck off, we have now to test only

$$6,\ 2,\ -2,\ -4,\ -8.$$

6 and -8 will be found to be the commensurable roots, and $x^2-2x+8=0$ the reduced quadratic, which has complex roots.

2. Find all the commensurable roots of

$$2x^5-31x^4+110x^3+95x^2-112x-64=0.$$

3. Find all the commensurable roots of

$$f(x)\equiv x^5-12x^4+25x^3-48x^2-26x+60=0.$$

Real roots lie between 12 and -2, and hence the divisors are 10, 6, 5, 4, 3, 2, 1, -1.

First we find $f(1)$ by the method of Synthetic Division:

1	-12	25	-48	-26	60
	1	-11	14	-34	-60
1	-11	$+14$	-34	-60	0

Hence $f(1)=0$, and $f(x)\equiv(x-1)(x^4-11x^3+14x^2-34x-60)$.

Having got 1 as one of the roots, we proceed further with

$$\phi(x)\equiv x^4-11x^3+14x^2-34x-60=0.$$

Now when we find $\phi(-1)$ by the above method, we get $\phi(-1)=0$ and $\phi(x)=(x+1)(x^3-12x^2+26x-60)$. Thus having got -1 as another root, we do the next process on the equation $\theta(x)\equiv x^3-12x^2+26x-60=0$.

We get $\theta(1)=-45$ and $\theta(-1)=-99$. Divisors 5 and 3 are rejected on account of $\theta(1)$; and 6, 4 are rejected on account of $\theta(-1)$; thus we have only to test 10 and 2.

It is found that one root is 10, and the other two roots are complex.

4. Solve the equation $3x^4-23x^3+35x^2+31x-30=0$. (*P. U. 1941*)

5. Solve completely, the equation $2x^4+x^3-2x^2-4x-3=0$, given that two of its roots are commensurable.

6. The roots of the equation

$$x^5-23x^4+160x^3-218x^2-257x-440=0$$

lie between -1 and 24. Obtain them. (*P. U. 1936*)

7. Find the roots of $16x^5+80x^4-96x^3+4x^2+23x-6=0$, given that the roots are all rational.

This is a case of multiple roots; the remark made at the end of Ex. 3, Examples IV (f) gives some indication of it, even though the repeated root is not an integer.

Here $$x^5+5x^4-6x^3+\frac{1}{2^2}x^2+\frac{1}{2^4}\cdot 23x-\frac{3}{2^3}=0,$$

and there is an indication that $\frac{1}{2}$ may occur 3 times as a root.

8. Solve the equation $81x^5-207x^4-9x^3+89x^2+2x-8=0$, given that three of the roots are of the form $\frac{k}{9}$, where k is an integer. (*P. U. 1920*)

9. Solve $864x^4+720x^3+210x^2+25x+1=0$, given that the roots are rational. (*P. U. 1921*)

If we proceed by making the coefficient of x^4 unity, we shall get -1, -2, -3, -4 as the roots of the transformed equation, and thus the roots of the given equation will be $-\frac{1}{12}$, $-\frac{1}{6}$, $-\frac{1}{4}$, $-\frac{1}{3}$.

Another method is of transformation to the reciprocal equation

$$y^4+25y^3+210y^2+720y+864=0,$$

$\left(\text{by putting } x=\frac{1}{y}\right)$. The required roots will be the reciprocals of the values of y.

10. Determine the integral roots of the equation

$$x^6+3x^5-36x^4-45x^3+93x^2+132x+140=0.$$

11. Determine the commensurable roots of the equation

$$6x^4-7x^3+8x^2-7x+2=0.$$

12. Factorize the expression

$$x^6+x^5-4x^4+4x^3+8x^2-32x-32.$$ [*Math. Trip. 1925*]

13. Solve completely the equation

$$x^8-57x^6+21x^5+756x^4-777x^3+20x^2+756x-720=0.$$

14. Find the integral roots of

$$x^5-47x^4+423x^3-140x^2+1213x+420=0.$$ (*Peterson*)

4·9. Horner's Method. This method consists in successive diminution of the roots of an equation by certain known numbers and possesses the immense advantage of giving both commensurable and incommensurable roots. The figures of any real root are obtained one by one, and we can approximate, therefore, with any degree of accuracy to the actual value of the root by continuing the process sufficiently. When the root is commensurable, we obtain first the figures which it contains in the integral part, and then all those which may

occur in the decimal part, and thus the root is found exactly; but when it is incommensurable, the figures in the decimal fraction will not terminate, and we shall have to content ourselves with a number of figures only. Such is the case when we find the value of an irrational number like $\sqrt{2}$; we cannot exactly find its value in decimal fraction, although we may find as many figures, as we like, in the decimal part. This being equivalent to the solution of the equation $x^2-2=0$, will be seen to be a particular case of Horner's Method.

When the commensurable root contains several figures, Newton's Method of Divisors will be found to be inconvenient, while Horner's Method will lead to the result by easy steps.

Since diminution of the roots of an equation can best be effected by the successive application of the method of Synthetic Division, as explained in Art. 3·4. Horner's method will be found to be the successive application of that method. In Ex. 9, Examples III (*d*), we diminish the roots of the equation $x^3-60x^2+281x-330=0$, first by 50 and then by 5, and are led to the conclusion that 55 is a root of the equation. This is a very simple case of Horner's method. We give below some examples, showing how this method can be used to evolve roots, by slow stages, in more difficult cases. As we proceed further, we shall explain a few more working rules which will be found very useful in the practical application of this important method.

Ex. 1. Find the positive root of the equation

$$x^3-208x^2-1059x-1278=0. \quad \ldots(1)$$

Descartes' Rule shows that the equation has one positive root, and since $f(200)$ is negative and $f(300)$ is positive, it follows that the only positive root lies between 200 and 300. It is not necessary to actually find the values of $f(200)$ and $f(300)$, since $f(200)$ is obviously negative, and $f(300)$ is seen to be positive after a little work. In this way the first figure of the root is found; sometimes a few trials will be necessary to locate the root so that the first figure may be got at. Diminishing the roots by 200, we get the transformed equation

$$x^3+392x^2+35,741x-533,078=0. \quad \ldots(2)$$

Now since the roots of (2) are less than the roots of (1) by 200, the positive root of (2) lies between 0 and 100. It is easily found to lie between 10 and 20; since the coefficients of the last and last but one term are sufficiently large in comparison with coefficients of x^2 and x^3, the former coefficients suggest the location of the root.

Diminishing the roots of (2) by 10, we get the next transformed equation $x^3+422x^2+43,881x-135,468=0. \quad \ldots(3)$

Again, since the roots of (3) are less than the roots of (2) by 10, and the positive root of (2) lies between 10 and 20, the positive root of (3) will lie between 0 and 10. The last two coefficients should here give a very good indication of the location of the root, since the figures to be tried now being small, the contributions of the terms containing x^2 and x^3 to the final positive or negative result will also be comparatively small. So, if we ignore the two leading terms of the equation, 3 gives a negative and 4 a positive result. In this way, the figure 3 is suggested. When we proceed to diminish the roots by 3, the absolute term of the next transformed equation, which is obtained at the very first operation of division, is found to be zero, showing that 3 is a root of equation (3). The roots of (1) having successively been diminished by 200, 10 and 3, the required positive root of (1) is 213.

The whole process may very well be shown in a compact form as below :—

1	−208	−1,059	−1,278 (213
	200	−1,600	−531,800 (
	−8	−2,659	−533,078
200		38,400	397,610
192		35,741	−135,468
200		4,020	135,468
392		39,761	0
10		4,120	
402		43,881	
10		1,275	
412		45,156	
10		1,284	
422		46,440	
3			
425			
3			
428			
3			
431			

In the above, a broken line separates the stage of each "diminution", and the numbers below it give the coefficients of the transformed equation. Thus the finally transformed equation is

$$x^2+431x^2+46,440x+0=0,$$

which means the quadratic $x^2+431x+46,440=0$, and its roots

are found to be -215 and -216. The other roots of the given equation, therefore, being greater than each of these by 213, are therefore, -2 and -3, respectively.

Ex. 2. Find the only positive root of the equation

$$f(x)\equiv 4x^3-21x^2+41x-102=0. \qquad \ldots(1)$$

It is easily seen that 6 is an upper limit. And here $f(5)$ is positive and $f(4)$ is negative, hence the root lies between 4 and 5. Diminishing the roots of (1) by 4, we get

$$4x^3+27x^2+65x-18=0. \qquad \ldots(2)$$

Since the positive root of (2) lies between 0 and 1 and is, therefore, < 1, $4x^3$ and $27x^2$ will be sufficiently small and we might get an approximate value of the root from the equation $65x-18=0$, obtained by neglecting $4x^3$ and $27x^2$ in (2).

Thus $x=\dfrac{18}{65}=0{\cdot}2\cdots$ and we can take $0{\cdot}2$ to be the number by which we further diminish the roots of (2). We get the next equation

$$4x^3+29{\cdot}4x^2+76{\cdot}28x-3{\cdot}888=0. \qquad \ldots(3)$$

As above, an approximate value of the root of (3) is given by $76{\cdot}28x-3{\cdot}888=0$, which gives $x=0{\cdot}05\cdots$

Diminishing the roots of (3) by $0{\cdot}05$, we find $0{\cdot}05$ to be a root of (3). Thus $4{\cdot}25$ is the required root of (1).

It should be noted that if $0{\cdot}2$ were too large a number by which we diminished the roots of (2), the next transformed equation (3) would have all the signs positive, showing that (3) had no positive root and that we had diminished the roots of (2) by taking too large a number.

We show the work below :—

4	-21	41	-102	$4{\cdot}25$
	16	-20	84	
	-5	21	-18	
	16	44	$14{\cdot}112$	
	11	65	$-3{\cdot}888$	
	16	$5{\cdot}56$	$3{\cdot}888$	
	27	$70{\cdot}56$	0	
	$0{\cdot}8$	$5{\cdot}72$		
	$27{\cdot}8$	$76{\cdot}28$		
	$0{\cdot}8$	$1{\cdot}48$		
	$28{\cdot}6$	$77{\cdot}76$		
	$0{\cdot}8$			
	$29{\cdot}4$			
	$0{\cdot}2$			
	$29{\cdot}6$			

When the decimal part of the root appears, we have to diminish the roots successively by decimal fractions, which make the work increasingly inconvenient. This inconvenience can easily be avoided if we transform the equation, after the integral part has been evolved at each successive stage by multiplying the roots by 10. Thus in the above, multiplying the roots of (2) by 10, we get the equation

$$4x^3+270x^2+6,500x-18,000=0,$$

and the root of this equation will approximately be 2, as above, and the "diminution", therefore, shall have to be done by the integral number 2. The next transformed equation will be obtained in the form

$$4x^3+\cdot 294x^2+7,628x-3,888=0.$$

This has a root between 0 and 1, and the roots should again be multiplied by 10, giving

$$4x^3+2,940x^2+762,800x-3,888,000=0,$$

which will have a root between 0 and 10, whose approximate value found as above, is 5 and the "diminution" is, therefore, to be done by 5. Thus the decimal fractions are done away with in the work which may now be exhibited as below :—

4	−21	41	−102	(4·25
	16	−20	84	
−5	21	−18000		
16	44	14122		
11	6500	−3888000		
16	556	3888000		
270	7056	0		
8	572			
278	762800			
8	14800			
286	777600			
8				
2,940				
20				
2,960				

Since, multiplying the roots by 10 is equivalent to annexing one cipher to the second coefficient, two ciphers to the third, and three ciphers to the fourth, and so on, (if the degree of the equation is higher than the third), we arrive at the simple rule :—

When the integral part of the root has been obtained, before the process of each subsequent "diminution", annex one cipher to the right of the coefficient in the first working column, two

ciphers to the right of the coefficient in the second working column, and so on.

Since the successive figures in the decimal part can still be obtained as before from an equation of the kind $ax+b=0$, whose terms containing x^2 and x^3 have been neglected, we have an easy rule for finding these figures. After ciphers have been annexed, the last coefficient divided by the last but one, will suggest the next figure. The last but one coefficient, for this reason, is known as "*trial-divisor.*" It will be seen that at every stage at which the trial-divisor suggests the right figure, the sign of the trial-divisor is opposite to that of the last coefficient. The reason for this lies in the fact that the figure being given by $ax+b=0$, a and b must be of opposite signs, for, otherwise the figure will be negative, which is possible only if the previous figure has been taken, by mistake, too large. In that case, the previous process of "diminution" must be set right.

We might, after two or three figures of the decimal part of an incommensurable root have been evolved, obtain the remaining figures with an approximation, simply by dividing the last coefficient by the last but one ; since x becomes at such a stage very small, the equation $ax+b=0$ will give a sufficiently good approximation.

4·91. Principle of the Trial Divisor. It will be clear from the worked out examples in the last Article that one of the most practical advantages of Horner's method is that after the second, or third figure (or even after the first, in some cases) of the root has been evolved, the transformed equation itself suggests by mere inspection the next figure of the root. Thus in Ex. 2, Art. 4·9, after the first diminution, the figure 2 is suggested by 18000/6500 correctly ; and after the second diminution the figure 5 is suggested by 38,88,000/7,62,800. The reason for this has been explained in the last Article. On account of the importance of the underlying principle, the following theorem may be given for the general case :—

Theorem. *If one of the roots of the equation*

$$p_0x^n+p_1x^{n-1}+\ldots+p_{n-1}x+p_n=0,$$

be small, say, between 0 *and* 1, *then an approximate value of that root is* $-p_n/p_{n-1}$.

For if x_1 denote a root of the equation in question we have

$$x_1=-\frac{p_n}{p_{n-1}}-\frac{x_1^2}{p_{n-1}}(p_{n-2}+p_{n-3}x_1+\ldots\ldots+p_0x_1^{n-2}).$$

Hence, if x_1 be small, we have approximately $x_1=-\dfrac{p_n}{p_{n-1}}$.

It is clear that the last two coefficients are of opposite signs.

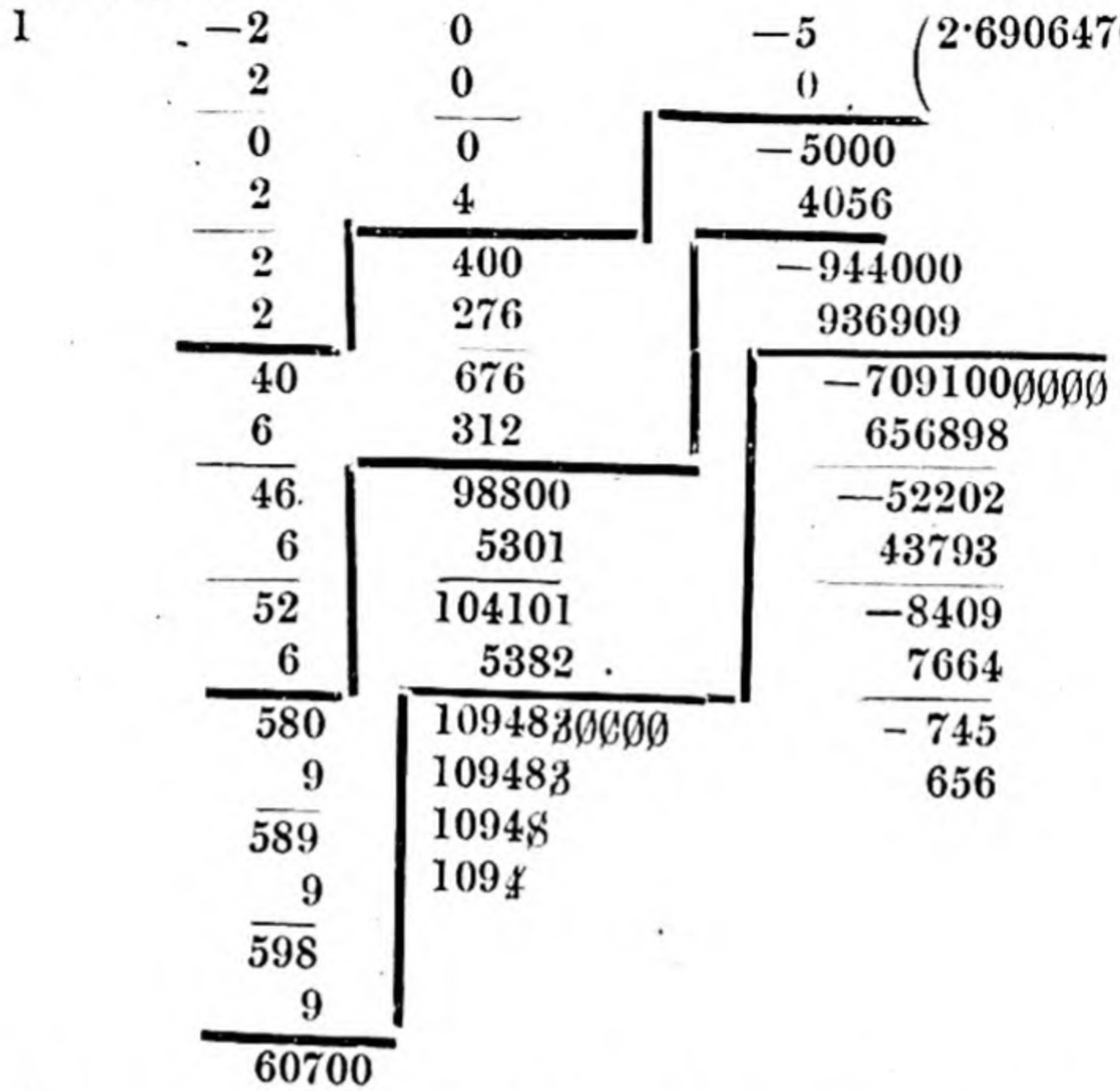
Ex. 3. Find the positive root of the equation $x^3-2x^2-5=0$ correct to seven places of decimals.

1	−2	0	−5	2·6906474
	2	0	0	
	0	0	−5000	
	2	4	4056	
	2	400	−944000	
	2	276	936909	
	40	676	−7091000000	
	6	312	6571165416	
	46	98800	−519834584000	
	6	5301	43823118144	
	52	104101	−81601465856000	
	6	5382	7669279332602 3	
	580	1094830000	−4908672529977000	
	9	364236		
	589	1095194236		
	9	364272		
	598	1095585080 0		
	9	2428736		
	60700	1095582795 36		
	6	2428752		
	60706	1095607082 8800		
	6	42503489		
	60712	1095611333 2289		
	6	42503538		
	607180	1095615583 582700		
	4			
	607184			
	4			
	607188			
	4			
	6071920			
	7			
	6071927			
	7			
	6071934			
	7			
	60719410			

Explanation.--By trial the root is seen to lie between 2 and 3. After diminishing the root by 2 and multiplying the roots of the resulting equation by 10, we get the equation with coefficients 1. 40, 400,—5000 ; since the second co-efficient is not sufficiently small in comparison with the third, the trial divisor does not suggest the right figure. If we take even 7 as the next figure, the effect of the second co-efficient is so great as to change the sign of the last term, showing that we have gone beyond the root and 7 is, therefore, too large. We must take care that the absolute term retains its sign throughout, after the first transformation. If we were to take a number too small (5 in this case), the error will show itself, just as in ordinary division by the next suggested figure being greater than 9. In the present case the right figure is 6. After operating with 6 and annexing zeroes, we see that the trial divisor becomes effective, and suggests the figure 9. After the operation with 9 and the usual annexation of zeroes to the co-efficients, we see that the trial divisor is greater than the absolute term. Zero is, therefore, the next figure. So we place zero after 9 and annex zeroes again to the right of the working columns before proceeding to the next operation.

Now the third co-efficient is quite large in comparison with the second, so that if we neglect the first and the second co-efficients altogether and obtain the remaining figures simply from the value of x given by $ax+b=0$, as pointed out in the last example, we can very well depend upon it for the next two or three figures for correctness. As we have now to perform simple division, an equal number of zeroes may be struck off from the divisor and the dividend, and after dividing by the next figure 6, we may employ the method of "contracted division". Thus instead of annexing a zero to the right of —52202, we strike off 3 from the right of 109483, but when dividing by 4, we mentally take into account the effect of 3. When dividing by the next figure 7, we have to mentally take into account the effect of 8, which gives 4 as the first figure in the product, since from 56 we should carry 6 not 5. The result correct to five places of decimals is 2·69065.

We perform below the work in full to compare with the result obtained above :—



Thus we see that the next three figures after zero have been obtained correctly.

Examples IV (*h*)

1. Find the only positive root of the equation

$$2x^3-21x^2-46x-50=0.$$

2. Find by the application of Horner's method the cube root of 1,953,125.

3. Find the positive root of the equation

$$20x^3-81x^2+141x-244=0.$$

4. Find the positive root of the equation $x^3-2x^2-3x-4=0$, correct to 4 decimal places.

5. Find, by Horner's process, the real root of $x^3-2=0$, correct to 5 decimal places. [*P. U. 1918*]

6. Find, by Horner's process, the cube root of 3, correct to 5 places of decimals. [*P. U. 1919 and 1924*]

If $x=(3)^{\frac{1}{3}}$, we have to solve the equation $x^3-3=0$ for the only real positive root.

7. Find to three decimal places the negative real root of the equation $x^3-x^2+12x+24=0$. [*P. U. 1931*]

8. Find by Horner's Method the real root of the equation $x^3-x^2-x-2000=0$ up to 3 places of decimals. (*P. U. 1941*)

9. Apply Horner's method to find to 3 places of decimals the root situated between 9 and 10 of the equation.

$x^4-3x^2+75x-10{,}000=0$ (*P. U. 1940 Burnside and Panton*)

4·92. Contraction applied to Horner's Method.

When the root is incommensurable and the result is to be obtained correct to several places of decimals, Horner's method, as used above, becomes very laborious. Though we may obtain two or three more correct figures, as in worked out Ex. 3 in the last Article, by division only, this simplification by itself does not go very far.

The leading coefficients may materially affect the trial divisor and the last coefficient even at the stage where we commence the process of division, and it is for this reason that out of all the figures of the quotient, we can rely only upon the first two or three. So if, instead of neglecting the leading coefficients altogether, we retain their important figures in their proper places, the result obtained will give all the figures correctly, with the exception of the last two or three. It is upon this principle that the contraction of Horner's method is effected, which is simply an extension of the contracted method of division. Thus at the stage at which the contraction begins, instead of annexing ciphers, we strike off one figure from the right of the trial-divisor, two figures from the right of the coefficient in the next preceding column, three figures from that in the next column, and so on. As in division, we shall, of course, mentally take into account the effect of the figures struck off. After all the figures in the column preceding that of the trial-divisor have been struck off, the process will reduce to that of ordinary division and can be continued until all the figures from the trial-divisor have been exhausted.

We illustrate the method of contraction by its application to the worked out example of § 4·91, the full process for which will enable us to institute a comparison.

The principle underlying the contraction may also be explained as below :—

After diminishing the roots by 6, we get the equation

$$x^3+60718x^2+1095558508x-519834584=0.$$

Now to multiply the roots of the equation by 10, we write $\frac{x}{10}$ for x. Thus we get the equation

$$0{\cdot}001x^3+607{\cdot}18x^2+109555850{\cdot}8x-519834584=0.$$

Now if we retain the integral parts of the coefficients, we have the equation

$$0.x^3+607x^2+109555850x-519834584=0.$$

It is to be noticed that the result of this is the same, as when instead of adding zeroes we cut off one figure from the second last, two from the third last, and so on. So we cut off all those which are not effective in contributing to the result. We shall work out the last example after the fifth approximation.

1	60718	109555850 8	−519834584	2·6906,474480
		2429	438233120	289
		109558280	−81601464	
		2429	76692791	
	607	10956070 9	−4908673	
		42	4382462	
		10956113	−526211	
		42	438246	
	6	10 9 3 6 1 3 3	−87965	
			87649	
			−316	
			219	
			−97	
			88	
			−9	
			9	

Ans. 2·6906474480289.

We strike off 8 from the right of the trial-divisor, 18 from the right of the coefficient in the next working column, and the next coefficient 1 goes altogether; the result is that only three co-efficients are left to be dealt with, as if our equation were a quadratic with coefficients 607, 109555850, −519834584. In multiplication and addition we do take into account the struck off figures, just as we do in contracted division. After diminution by 4, we strike off 9 from the trial-divisor, 07 from the next coefficient and proceed to diminish the roots by 7 in the same way. At the next stage the working column next to that of trial-divisor goes away altogether and the rest of the work proceeds as in contracted division. Thus we get easily 13 figures

in the decimal part out of which we may rely upon the first 10. Hence the result correct to 10 places of decimals is 2·6906474480.

The step from which the contracted process should commence depends upon the number of decimal places to which the root is required. The number of figures obtained, after the contracted method begins, will be seen to be one less than the number of figures in the trial divisor.

Examples IV (*i*)

1. Find the positive root of the equation $x^5-4x-2=0$ correct to seven places of decimals. [*P. U. 1917*]

2. An equation has its absolute term negative and possesses only one positive root. Explain why throughout the transformations, by Horner's process, the absolute term must retain its sign. Point out all those examples, illustrating this fact, which have already been done.

3. Obtain by Horner's method the largest of the roots of the equation $x^3-3x^2-2x+5=0$, to three places of decimals, and find the integral parts of the other roots.

[*Todhunter, P. U. 1927*]

The roots are easily seen to lie (*i*) between -2 and -1, (*ii*) between 1 and 2, (*iii*) between 3 and 4.

When for the largest root the diminution is done by 3, the absolute term will be found to change from positive to negative. The reason is that the transformed equation will have only one root between 0 and 1 and hence the value 0 should make the polynomial negative which is possible only when the absolute term is negative. Subsequently, the negative sign must be retained throughout.

4. Find the second positive and the negative root of the above equation $x^3-3x^2-2x+5=0$, each correct to 10 decimal places.

For the positive root, between 1 and 2, as successive diminutions are done, the absolute term will be found to retain the positive sign throughout. The reason is that both the roots keep above zero in all the transformed equations, and hence $f(\infty)$ being positive, $f(0)$ must also be positive for all the equations.

To find the negative root, the best way is to change the signs of the roots, so that the negative root here becomes the positive root of the equation $x^3+3x^2-2x-5=0$.

5. Find the sum of the three roots of the equation

$$x^3+3x^2-2x-5=0,$$

from the results obtained in Exs. 3 and 4 above. Explain why it differs from -3, which is the exact sum found otherwise.

6. Find the root of the equation $x^4-x^3-x^2-8x+9=0$, which lies between 2 and 3. [*P. U. 1928*]

On diminishing the roots by 2, the absolute term will be found to change its sign, showing that we have passed over a root, *i.e.*, another positive root lies between 0 and 2 ; which is, in fact, 1.

7. Find the root of the equation $x^5+6x^3+5x^2+10x-18=0$, to six places of decimals.

8. Obtain to 7 places of decimals the root of the equation $x^4-12x+7=0$, which lies between 2 and 3. [*P. U. 1925*]

9. Find the cube root of 25 to 10 places of decimals by Horner's method. [*P. U. 1921*]

10. Find the real roots of the equation

$$x^4+9x^3+16x^2+23x-80=0.$$

11. Find the positive root of $x^5-4x-2000=0$.

12. Find the positive root of Newton's cubic $x^3-2x-5=0$.

13. Find the positive root of $x^3+2x^2-5x-7=0$.

14. Find the positive root of $2x^3+x^2-15x-59=0$. (*Fine*)

15. Find the positive roots of the equation

$$x^4+x^3-3x^2-x-4=0.$$ (*Nelto*)

4·93. Applications of Horner's Method to equations having nearly equal roots.

When two or more roots are nearly equal, the trial divisor cannot give any indication of the next figure of the roots until the roots have been separated. For, suppose two (or more) roots of the equation are nearly equal to h but greater than h. If we diminish the roots of the equation $f(x)=0$ by h, we get the transformed equation $f(x+h)=0$,

or $$f(h)+xf'(h)+\frac{x^2}{2!}f''(h)+\ldots\ldots+\frac{x^n}{n!}f^n(h)=0.$$

By the rule of the trial divisor the next figure of the root is $-\frac{f(h)}{f'(h)}$, but since two (or more) roots are nearly equal to h, $f(h)$ and $f'(h)$ are both small and the ratio $-\frac{f(h)}{f'(h)}$ is almost indeterminate. So we shall have to find by trial, the location of the roots until we have succeeded in separating them.

Again, in diminishing the roots of the transformed equation, we choose the greatest number which, in the process of transformation, will keep the sign of the absolute term intact, for otherwise we pass over a root. When an even number of roots

are nearly equal, lying between h and $h+1$, the above test fails, for in passing over an even number of roots, the absolute term will retain its sign, but will change its sign if we pass over an odd number of roots. The observations made in the following examples will show the methods which may be helpful.

Examples IV (j)

1. Given that two roots of the equation $4x^3-10x^2+3x+4=0$ are positive, show that they lie between 1 and 2. Also separate them.

Here $f(1)=1$ and $f(2)=2$, which are both positive, so that we have not the same test to guide us in our search for the proper figures of the roots as in the previous cases, *viz.*, the change of sign. We can, however, discover this fact by an indirect method. If we diminish the roots by 2, we get the equation $4x^3+14x^2+11x+2=0$, showing that this equation has no positive root, *i.e.*, the original equation has its roots between 0 and 2. If, however, we diminish by 1, we get the equation $4x^3+2x^2-5x+1=0$, which has two variations of signs as the original equation; hence the two positive roots lie between 1 and 2. After diminishing the roots by 1, we multiply the roots of the transformed equation by 10, so we get the equation

$$4x^3+20x^2-500x+1000=0, \text{ or } x^3+5x^2-125x+250=0,$$

the roots of which lie between 0 and 10. They are seen to lie between 2 and 3 ; 7 and 8. The roots are now separated and we can proceed on as in the previous cases.

An indication of the approximate equality of the roots of the equation $f(x)=0$ is also afforded by the fact, that the values of $f(x)$ are nearly equal for the values of the consecutive integers between which the roots lie. Taking the case in hand, the superior limit of the positive roots is 3. Now $f(0)=4$, $f(1)=1$, $f(2)=2$, $f(3)=31$. So there is a probability (not certainty) of equal roots lying between 1 and 2.

2. The equation $x^3-7x+7=0$ has two roots between 1 and 2 ; find each of them to eight decimal places.

3. Find the nearly equal roots of the equation

$$x^3+x^2-10x+9=0. \qquad (\textit{Fine})$$

4. Find the two nearly equal roots of the equation $x^3-49x^2+658x-1379=0$, which lie between 23 and 24.

(*Burnside and Panton*)

It may be pointed out, that so long as the the two roots remain together, a guide to the proper figure of the root may be obtained by dividing twice the last coefficient by the second last, or the second last by twice the third last. The reason is that

the proposed equation approximates now to the quadratic formed by the last three terms in each transformed equation.

(*Burnside and Panton*)

For, if the transformed equation be

$$p_0x^n+p_1x^{n-1}+\ldots\ldots+p_{n-2}x^2+p_{n-1}x+p_n=0,$$

then $p_{n-2}x^2+p_{n-1}x+p_n=-x^3(p_{n-3}+p_{n-4}x+\ldots+p_0x^{n-3})$.

If x be small, we can neglect its cube and higher powers, so that the equation approximates to $p_{n-2}x^2+p_{n-1}x+p_n=0$.

If the equation has two nearly equal roots α, then

$$\alpha=\frac{-p_{n-1}}{2p_{n-2}}, \text{ or } \frac{-2p_n}{p_{n-1}}$$

approximately. We shall also have an indication of the separation of the roots when the numbers suggested in this way by the last three coefficients are different, *i.e.*, when $\frac{p_{n-1}}{2p_{n-2}}$ and $\frac{2p_n}{p_{n-1}}$ are different. An indication of the approximate equality of roots can also be inferred from the fact that the equation of the squared differences will have its absolute term very small.

5. The equation $x^4-14x^2+16x+9=0$ has two roots between 2 and 3. Find them to two places of decimals. [*P. U. 1937*]

6. Calculate to three decimal places each of the two roots lying between 4 and 5 of the equation

$$x^4+8x^3-70x^2-144x+936=0. \quad (\textit{Burnside and Panton})$$

7. Solve the equation $20x^3-30x^2+12x-1=0$ all of whose roots lie between zero and unity. (*Gauss*)

If an equation has three nearly equal roots in the interval (α, β), the value of one of them being $\alpha+h$, we diminish by α. If the transformed equation be

$$p_0x^n+\ldots+p_{n-3}x^3+p_{n-2}x^2+p_{n-1}x+p_n=0,$$

then x being small, the equation approximates to

$$p_{n-3}x^3+p_{n-2}x^2+p_{n-1}x+p_n=0,$$

which is nearly the same as $p_{n-3}(x-h)^3=0$.

Hence $h=\frac{p_{n-2}}{-3p_{n-3}}=-\frac{p_{n-1}}{p_{n-2}}=\frac{-3p_n}{p_{n-1}}$.

8. The equation $70x^4-140x^3+90x^2-20x+1=0$ has all its roots real, which lie in the interval (0, 1); find them to seven decimal places.

(*Gauss*)

CHAPTER V

SOLUTION OF CUBIC AND BIQUADRATIC EQUATIONS

5·1. The algebraic (or general) solution of an equation consists in finding accurate algebraic expressions, depending on the coefficients of the equation, which when substituted for the unknown quantity render both sides identical; these expressions are the general values of the roots, *e.g.*, $(-b \pm \sqrt{b^2-4ac})/2a$ is the algebraic solution of the quadratic $ax^2+bx+c=0$. The algebraic solutions of the simple and the quadratic equations being already known to the student, the algebraic and graphical solutions of the cubic and the biquadratic equations will be given in this Chapter. It may be noted that the impossibility of the existence of the algebraic solutions for general equations of degree higher than the fourth was conclusively established first by Abel, and afterwards by Wantzel. For the fifth degree equation, a general solution involves elliptic integrals and is, therefore, not algebraic but transcendental.

5·2. Graphical representation of a polynomial.

Let $f(x) \equiv a_0x^n + a_1x^{n-1} + \ldots\ldots + a_{n-1}x + a_n = 0$ be the given polynomial.

Put $$y = f(x).$$

For each value of x, the corresponding value of $f(x)$, *i.e.*, y can be found by the method of Art. 1·31. If we now take two perpendicular directed lines with a common zero point as the axes of co-ordinates, each set of values of (x, y) will respresent a point referred to the co-ordinate axes already chosen. The points so plotted form a **discontinuous graphical representation** of the function $f(x)$. These points do not lie at random but suggest a curve which runs through them smoothly from left to right.

The greater the number of points plotted, the more accurate will be the graphical representation, but in every case it will be an **approximate** graph of the **accurate** curve.

Useful information is also afforded by the derivative $f'(x)$ of $f(x)$. The vanishing of $f'(x)$ at any point of the curve signifies that the tangent to the curve at the point is parallel to the

x-axis. Such values of x for which $f'(x)=0$ are in general, the bend points on the curve. If in passing through such a value, $f'(x)$ changes sign from $+ve.$ to $-ve.$, the bend is like that as shown in Fig. (i). If the change is from $-ve.$ to $+ve.$, the bend is like that of Fig. (ii). For the sake of simplicity, suppose that $f(x)$, $f'(x)...f^{(m-1)}(x)$ all vanish at the origin. The equation of the curve will then become

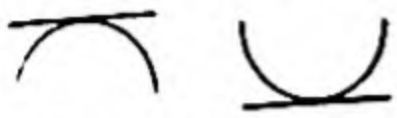


Fig. (i) Fig. (ii)

$$y=ax^m+bx^{m+1}+.....+kx^n=0.$$

The tangent at the origin is the x-axis. For sufficiently small numerical values of x, y has the same sign as ax^m. If m is even, the points of the curve in the vicinity of the origin lie on the same side of the x-axis. If m is odd, the points with small negative abscissae lie on one side of the x-axis and those with small positive abscissae lie on the opposite side. Thus if m is even, the curve in the vicinity of the origin is a U-shaped curve and there is a bend at the origin. If m is odd, the curve crosses the tangent at the point and is called the point of inflexion.

EXAMPLES V (a)

1. Draw the graphs of the following curves in the same figure:

$$y=x,\ y=x^3,\ y=x^5,\ y=x^7.$$

2. Draw the graphs of the following curves in the same figure:

$$y=x^2,\ y=x^4,\ y=x^6,\ y=x^8.$$

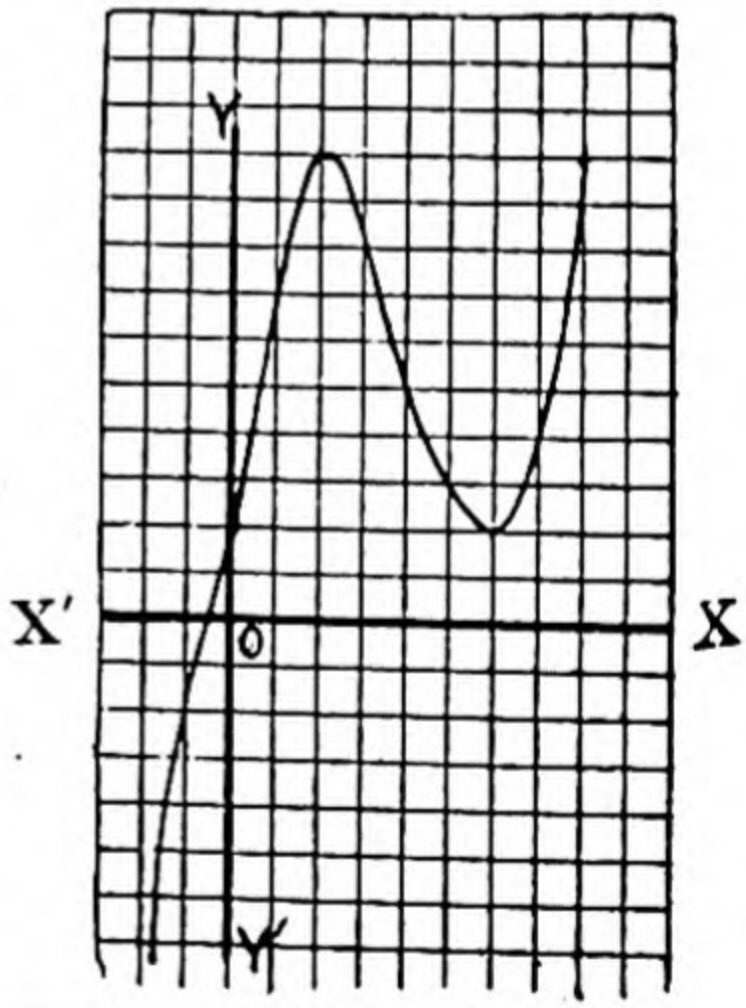
3. Draw the graph of

$$x^3-6x^2+9x+1=0$$

Solution : $f(x)=x^3-6x^2+9x+1.$

$$\tfrac{1}{3}f'(x)=x^2-4x+3,$$

and this vanishes for $x=1$, $x=3$, and for these values of x, $f''(x)$ does not vanish. The curve $y=x^3-6x^2+9x+1$ has two bend points whose abscissae are $x=1$, $x=3$.



The following table gives some of the important points on the curve :

x	$-\infty$	-1	0	1	3	∞
y	$-\infty$	-15	1	5	1	∞

We now trace the curve, the unit of length being twice the side of the square.

The equation has, therefore, only one real root and it is negative, for the curve cuts the x-axis only once, and on the left of y-axis.

4. Draw the graphs of the following functions :—

(*i*) $2x^3-3x^2-12x+13$,

(*ii*) $4x-\frac{1}{3}x^3$,

(*iii*) $x^3-9x^2+15x+13$,

(*iv*) $15+9x-3x^2-x^3$,

(*v*) $\frac{1}{3}x^3-x^2-3x+1$.

5·21. Graphical Solution of Equations. Let $y=f(x)$, $y=\phi(x)$ be the equations of two rational integral curves. The abscissae of the points of intersection of the curves are the roots of the equation

$$f(x)=\phi(x).$$

This suggests the graphical solution of equations by going through the steps indicated above. The method then consists in constructing two convenient curves the abscissae (or ordinates) of whose points of intersection are the roots of the given equation. The ingenuity lies in constructing such curves as can be traced easily.

5·22. Graphical Solution of the Cubic Equation. It has been proved that the general cubic equation $a_0x^3+3a_1x^2+3a_2x+a_3=0$, can be reduced to the form $x^3+3\mathrm{H}x+\mathrm{G}=0$. It is therefore, sufficient to consider the equation

$$x^3+3\mathrm{H}x+\mathrm{G}=0.$$

The equation gives the abscissae of the points of intersection of the line $y=-3\mathrm{H}x-\mathrm{G}$, and the cubic curve, $y=x^3$, which has a point of inflexion at the origin.

5·23. Solution of a Cubic (*Continued*). Another convenient method is the construction of two conics which have a known common point. Two conics intersect in four points. If one of these points be known, the abscissae of the remaining three points will be the roots of the given equation.

Let the equation be $x^3+3\mathrm{H}x+\mathrm{G}=0$ which can be written as

$$x^4+3\mathrm{H}x^2+\mathrm{G}x=0,$$

so that one zero-root has been introduced. Writing the equation as

$$x^4+x^2+(3\mathrm{H}-1)x^2+\mathrm{G}x=0,$$

it is seen that the roots of this equation are the abscissae of the points of intersection of the parabola.

$$y=x^2,$$

and the circle $\quad x^2+y^2+(3\mathrm{H}-1)y+\mathrm{G}x=0,$

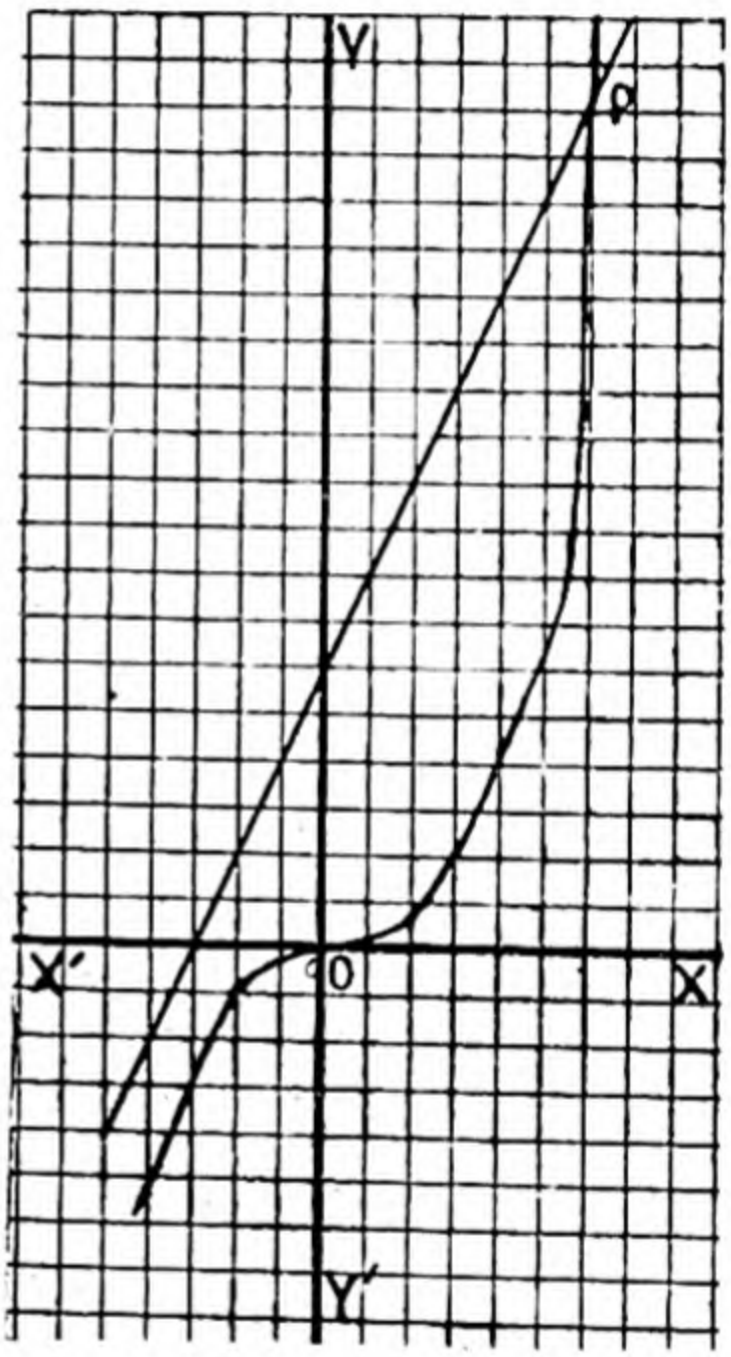
so that the roots of the given equation are the abscissae of the points of intersection, other than the origin, of the above circle and the parabola.

EXAMLES V (*b*)

1. Solve the equation $x^3-6x-9=0$.

Writing the equation as $\frac{x^3}{3}=2x+3$, we see that the roots of the equation are the abscissae of the points of intersection of the curves

$$y=\tfrac{1}{3}x^3, \text{ and } y=2x+3.$$



We accordingly trace the curves as in the adjoining figure. The unit of length chosen is twice the side of the square. It is then seen that the curves intersect at the point whose abscissa is 3. The other two roots are imaginary, as the cubic curve does not intersect the line in any other real point.

2. Solve the equation

$$x^3-7x-6=0,$$

The equation can be written as

$$x^4+x^2-8x^2-6x=0.$$

which differs from the original equation in having an additional root zero. The roots of this equation are the abscissae of the points of intersection of

$$y=x^2 \text{ and } x^2+y^2-8y-6x=0.$$

The first is a parabola, and the second is a circle with centre (3, 4) and radius 5. Draw the two curves (scale : 1 unit=length of two squares). Besides the origin the curves intersect in points whose abscissae are -1, -2, 3. [See Figure on p. 148.]

These are, therefore, the roots of the equation.

3. Solve graphically the equations :—

(*i*) $x^3-7x+6=0$,
(*ii*) $x^3-3x-2=0$,
(*iii*) $x^3-6x-9=0$.

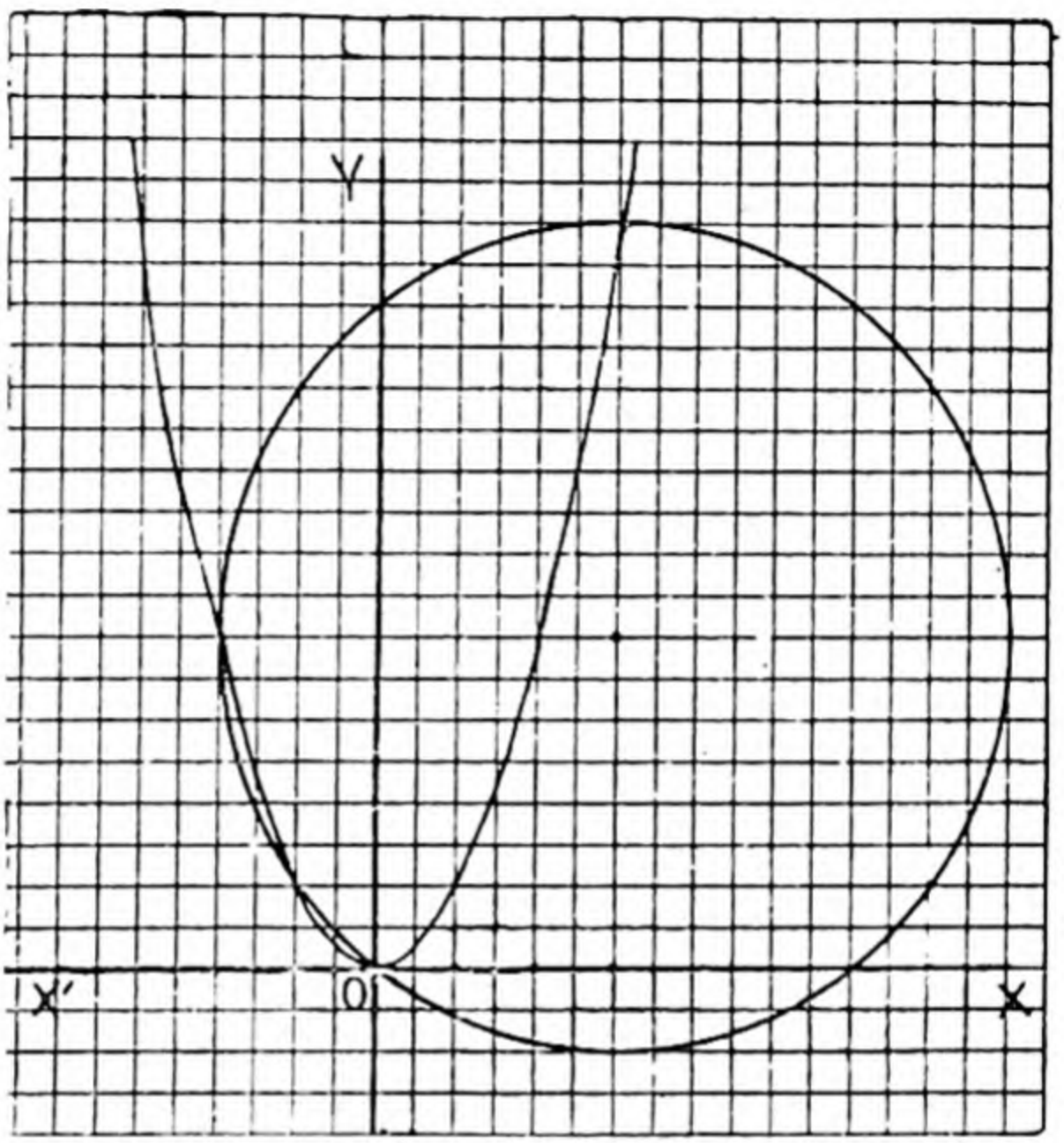
4. Determine the values of a such that the equation $x^3+12x+a=0$ has only one real root. (*Math. Trip. 1923*)

Solution. The equation of the tangent to $y=x^3$ at (x', y') is

$$y-y'=3x'^2(x-x'), \text{ or } y=3x'^2x-2x'^3.$$

If this line be parallel to $y=12x$, then $3x'^2=12$, or $x'=\pm 2$.

Thus the equations of the tangents to $y=x^3$, parallel to $y=12x$, are $y=12x\pm 16$.



Hence if a lies between ± 16, there are three real roots and all distinct; if $a=\pm 16$ there are three roots, of which two are equal; and if a does not lie between ± 16, there is only one real root.

5. Discuss the nature of the root of the cubic

$$x^3+3\mathrm{H}x+\mathrm{G}=0.$$

53. Cardan's Solution of the Cubic Equation.

Let the equation, $a_0x^3+3a_1x^2+3a_2x+a_3=0$, be reduced to the form $z^3+3\mathrm{H}z+\mathrm{G}=0$, where $z\equiv a_0x+a_1$, $\mathrm{H}\equiv a_0a_2-a_1^2$, and $\mathrm{G}\equiv a_0^2a_3-3a_0a_1a_2+2a_1^3$.

Assume $$z=u+v,$$

then $$z^3-3uvz-(u^3+v^3)=0.$$

Now comparing the coefficients, we get

$$uv=-\mathrm{H}.\quad u^3+v^3=-\mathrm{G}.$$

Hence u^3, v^3 are the roots of the t-quadratic equation

$$t^2+\mathrm{G}t-\mathrm{H}^3=0,$$

whence we obtain

$$u^3=(-\mathrm{G}+\sqrt{\mathrm{G}^2+4\mathrm{H}^3})/2,\quad v^3=(-\mathrm{G}-\sqrt{\mathrm{G}^2+4\mathrm{H}^3})/2.$$

Substituting for v its value, $\frac{-H}{u}$, we have $z=u+\frac{-H}{u}$ as the algebraic solution of the equation $z^3+3Hz+G=0$.

As the cube root of u^3 has the three values, u. ωu, $\omega^2 u$, (ω being a cube root of unity) we obtain three values of z, *viz.*

$$u+\frac{-H}{u},\ \omega u+\omega^2.\frac{-H}{u},\ \omega^2 u+\omega.\frac{-H}{u}.$$

Having known z. the corresponding values of x can be found from the relation $z\equiv a_0 x+a_1$.

5·31. Discriminant of the Cubic $x^3+3Hx+G=0$.

Let α, β, γ be the roots, then

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix} =(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha).$$

$$\therefore\ (\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2$$

$$= \begin{vmatrix} 3 & \alpha+\beta+\gamma & \alpha^2+\beta^2+\gamma^2 \\ \alpha+\beta+\gamma & \alpha^2+\beta^2+\gamma^2 & \alpha^3+\beta^3+\gamma^3 \\ \alpha^2+\beta^2+\gamma^2 & \alpha^3+\beta^3+\gamma^3 & \alpha^4+\beta^4+\gamma^4 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 0 & -6H \\ 0 & -6H & -3G \\ -6H & -3G & 18H^2 \end{vmatrix} = -27 \begin{vmatrix} -1 & 0 & 2H \\ 0 & 2H & G \\ 2H & G & -6H^2 \end{vmatrix}$$

$$=-27(G^2+4H^3).$$

5·32. Nature of the Roots of the Cubic.

By art. 5·3, the roots of the equation

$$x^3+3Hx+G=0,$$

are $$x_1=u+v,\ x_2=wu+w^2v,\ x_3=w^2u+wv,$$

or $$u+v,\ -\tfrac{1}{2}(u+v)\pm\tfrac{1}{2}(u-v)\sqrt{-3},$$

where $$v=\frac{-H}{u}.$$

1. If $G^2+4H^3=0$, u and v are equal, and $x_2=x_3$. The cubic, therefore, has two equal roots. Conversely, if two roots are equal, $G^2+4H^3=0$. For, three pairings of the roots $x_1=x_2$, $x_1=x_3$, $x_2=x_3$ give, respectively,

$$u=w^2v,\ u=wv,\ u=v,$$

and, in each case, $u^3=v^3$. Thus the t-quadratic of art. 5·3 has equal roots, which gives $G^2+4H^3=0$.

Otherwise, from the last art., if

$$\alpha=\beta, \text{ or } \beta=\gamma, \text{ or } \gamma=\alpha, \quad G^2+4H^3=0.$$

2. If $G=0$, $H=0$, the cubic has three equal roots, and conversely. For, $x_2=x_3$ and $x_1=x_2$ respectively give $u=v$, $u=\omega v$, *i.e.*, $v=\omega v$. Consequently, $v=0$ and therefore, $u=0$. The t-quadratic has, therefore, two zero roots. Hence

$$G=0, \quad H=0.$$

3. If $G^2+4H^3>0$, u and v are real and x_1 is real, but x_2, x_3 are conjugate imaginaries. Conversely, if x_2, x_3 are complex, u and v are real and the t-quadratic gives

$$G^2+4H^3>0.$$

4. If $G^2+4H^3<0$, u^3 and v^3 are conjugate imaginaries, and so will be u and v.

Let $$u=a+ib, \quad v=a-ib.$$

Then $$x_1=2a, \quad x_2=-a-b\sqrt{3}, \quad x_3=-a+b\sqrt{3},$$

so that all the roots are real. Conversely, if all the roots are real, $u+v$ must be real and $u-v$ pure imaginary. Consequently u and v are conjugate complex and, therefore, u^3 and v^3 which are the roots of t-quadratic are conjugate imaginary. Hence $G^2+4H^3<0$.

5·33. For a discussion of the nature of the roots of the cubic $z^3+3Hz+G=0$, the following method may also be noticed :—

Let the roots be -2α, $\alpha \pm \sqrt{\gamma^2}$; then the sum of the products of the roots in pairs

$$=-3\alpha^2-\gamma^2=3H, \qquad \ldots (i)$$

and the product of the roots $=-2\alpha(\alpha^2-\gamma^2)=-G.$ $\qquad \ldots (ii)$

Eliminating α^2, we get

$$\gamma^2=-\frac{27}{4}\frac{G^2+4H^3}{(4\gamma^2+9H)^2}.$$

(*i*) If the roots are all real, γ^2 is positive hence G^2+4H^3 is negative,

and conversely.

(*ii*) If two roots are complex, γ^2 is negative, so G^2+4H^3 is positive,

and conversely.

(*iii*) If two roots are equal, $\gamma^2=0$, hence $G^2+4H^3=0$, and conversely.

(*iv*) If all the roots are equal, $\alpha=0$ and $\gamma=0$, therefore $H=0$, $G=0$, separately, in $G^2+4H^3=0$, and conversely.

5·34. Solution of the cubic by expressing it as the sum or difference of two cubes.

Assume

$$a_0x^3+3a_1x^2+3a_2x+a_3\equiv \mathrm{L}\,(x-\alpha)^3-\mathrm{M}\,(x-\beta)^3,\ldots \quad \ldots \quad (1)$$

where L, M, α, β, are the numbers to be determined.

Comparing the coefficients on the two sides, we get

$$a_0=\mathrm{L}-\mathrm{M}, \quad \ldots \quad (2)$$
$$a_1=-\mathrm{L}\alpha+\mathrm{M}\beta, \quad \ldots \quad (3)$$
$$a_2=\mathrm{L}\alpha^2-\mathrm{M}\beta^2, \quad \ldots \quad (4)$$
$$a_3=-\mathrm{L}\alpha^3+\mathrm{M}\beta^3. \quad \ldots \quad (5)$$

Multiply (2), (3), (4), each by α, and add, respectively to (3), (4), (5) ; then.

$$a_0\alpha+a_1=-\mathrm{M}(\alpha-\beta),$$
$$a_1\alpha+a_2=\mathrm{M}\beta(\alpha-\beta),$$
$$a_2\alpha+a_3=-\mathrm{M}\beta^2(\alpha-\beta).$$

Hence $(a_0\alpha+a_1)(a_2\alpha+a_3)=(a_1\alpha+a_2)^2.$

Similarly $(a_0\beta+a_1)(a_2\beta+a_3)=(a_1\beta+a_2)^2.$

Thus α and β are the roots of the equation

$$(a_0x+a_1)(a_2x+a_3)=(a_1x+a_2)^2. \quad \ldots \quad (6)$$

Having determined α, β, from (6), L and M can be found from (2) and (3). Hence the given equation is thrown in the form

$$\mathrm{L}\,(x-\alpha)^3-\mathrm{M}\,(x-\beta)^3=0.$$

Therefore $$\left(\frac{x-\alpha}{x-\beta}\right)^3=\frac{\mathrm{M}}{\mathrm{L}},$$

i.e. $$\frac{x-\alpha}{x-\beta}=\left(\frac{\mathrm{M}}{\mathrm{L}}\right)^{\frac{1}{3}},\left(\frac{\mathrm{M}}{\mathrm{L}}\right)^{\frac{1}{3}}\omega,\ \left(\frac{\mathrm{M}}{\mathrm{L}}\right)^{\frac{1}{3}}\omega^2, \text{ respectively,} \quad \ldots \quad (7)$$

from which x can be found.

5·35. If $f\,(x,\,y)=a_0x^3+3a_1x^2y+3a_2xy^2+a_3y^3$, the equation (6) can be written as

$$\frac{\delta^2f}{\delta x^2}\cdot\frac{\delta^2f}{\delta y^2}=\left(\frac{\delta^2f}{\delta x\delta y}\right)^2,$$

where y is put equal to unity after differentiation. This equation is called the *Hessian quadratic* of the cubic.

From Arts. 5·3 and 5·34, it follows that the solution of a cubic equation can be made to depend on the solution of equations of degree not higher than the second.

Examples V (*c*)

1. Solve the equation $28x^3-9x^2+1=0$. (*P. U. 1924, 1940*)

Put $z\equiv 28x-3$; the reduced equation is $z^3-27z+730=0$.

Let $z=u+v$, then $z^3-3uvz-(u^3+v^3)=0$,

therefore, $uv=9$, $u^3+v^3=-730$,

and so u^3 and v^3 are the roots of $t^2+730\,t+729=0$,

so $$u^3=-1,\ v^3=-729.$$
$$\therefore\quad u=-1,\ -\omega,\ -\omega^2.$$

Hence $28x-3\equiv z=-10,\ -\omega-9\omega^2,\ -\omega^2-9\omega$, respectively,

$$i.e.,\quad x=-\frac{1}{4},\ \frac{2\pm\sqrt{-3}}{7}.$$

NOTE 1. Having known $x=-\frac{1}{4}$, it is more convenient to remove the factor $4x+1$ from $28a^3-9a^2+1$; then the remaining roots are the roots of the equation $7x^2-4x+1=0$.

NOTE 2. This equation can also be reduced to the standard form by writing $x=\frac{1}{y}$; then $y^3-9y+28=0$.

2. Solve the equation $7x^3-21x^2+231x-7=0$. *(P. U. 1922)*

Removing the extraneous factor 7, we assume

$$x^3-3x^2+33x-1\equiv \mathrm{L}(x-\alpha)^3-\mathrm{M}(x-\beta)^3. \quad \ldots\ (1)$$

Comparing coefficients on both sides, we get

$$1=\ \mathrm{L}-\mathrm{M}, \quad \ldots\ (2)$$
$$-1=-\mathrm{L}\alpha+\mathrm{M}\beta, \quad \ldots\ (3)$$

etc.

Also α and β are the roots of the equation

$$(x-1)(11x-1)=(-x+11)^2,$$

or $$x^2+x-12=0,$$
$$\therefore\quad \alpha=-4,\ \beta=3.$$

From (2) and (3), we get

$$\mathrm{L}=\frac{2}{7},\qquad \mathrm{M}=\frac{-5}{7}.$$

Hence the cubic reduces to

$$\frac{2}{7}(x+4)^3+\frac{5}{7}(x-3)^3=0.$$

$$\therefore\quad \left(\frac{x+4}{x-3}\right)^3=\frac{-5}{2},$$

or $$\frac{x+4}{x-3}=-\left(\frac{5}{2}\right)^{\frac{1}{3}},\ -\left(\frac{5}{2}\right)^{\frac{1}{3}}\omega,-\left(\frac{5}{2}\right)^{\frac{1}{3}}\omega^2.$$

$$\therefore\quad x=\frac{3\sqrt[3]{5}-4\sqrt[3]{2}}{\sqrt[3]{2}+\sqrt[3]{5}},\ \frac{3\omega\sqrt[3]{5}-4\sqrt[3]{2}}{\sqrt[3]{2}+\omega\sqrt[3]{5}},\ \frac{3\omega^2\sqrt[3]{5}-4\sqrt[3]{2}}{\sqrt[3]{2}+\omega^2\sqrt[3]{5}}.$$

3. Solve the following cubic equations :—

(*i*) $x^3-27x+54=0$,

(*ii*) $x^3-6x-9=0$,

(*iii*) $x^3-12x-65=0$,
(*iv*) $2x^3+3x^2+3x+1=0$, (*P. U. 1941 Supp.*)
(*v*) $x^3-15x^2-357x+5491=0$, (*Selwyn*) (*P. U. 1938*)
(*vi*) $27x^3+54x^2+198x=73$, (*Corpus*)
(*vii*) $x^3+6x^2+9x+4=0$, (*P. U. 1934*)
(*viii*) $x^3-6x^2-6x-2=0$,
(*ix*) $x^3-6x^2-12x+112=0$,
(*x*) $4x^3+9x^2+18x+17=0$,
(*xi*) $x^3+12x^2-6x+10=0$. (*P. U. 1932*)

4. Find the relation between q and r, in order that the equation $x^3+qx+r=0$ may be put in the form

$$x^4=(x^2+ax+b)^2.$$

Hence solve the equation $8x^3-36+27=0$. (*Hall & Knight*)

5, Determine, for what range of numerical values of k, the roots of $2x^3-9x^2+12x-k=0$ are real. (*Math. Trip.*)

6. Show that

$$x^3+u^3+v^3-3xuv\equiv(x+u+v)(x+\omega u+\omega^2 v)(x+\omega^2 u+\omega v).$$

Hence solve the equation $x^3+3\mathrm{H}x+\mathrm{G}=0$.

Apply this method to solve $x^3-3ax+(a^3+1)=0$.

7. Solve completely, $x^3-3abx+a^3+b^3=0$. (*P. U. 1943*)

8. If $z^3+3hz+g=\dfrac{1}{\mu-\nu}[\mu(z+\nu)^3-\nu(z+\mu)^3]$ for all values of z, show that

$$2\mu=\frac{g}{h}+\frac{\sqrt{\Delta}}{h},\quad 2\nu=\frac{g}{h}-\frac{\sqrt{\Delta}}{h},\text{ where }\Delta=g^2+4h^3.$$

Show that the cubic $4z^3-27a^2(z+a)=0$ has two equal roots. (*Math. Trip.* I, 1927)

5·36. Equal roots of the Hessian. When the roots of the Hessian quadratic are equal, it is possible to put the cubic as the difference of two cubes only if all the three roots of the cubic are equal. The equal roots of the cubic are equal to the equal roots of the Hessian. This is obvious from (1) Art. 5 34.

When all the roots of the cubic are not equal, the method fails, so far as the question of putting the cubic as the difference of two cubes is concerned. The Hessian quadratic will, however, be a factor of the cubic and thus the roots can easily be found. We give the proof below.

Let the cubic be reduced to $x^3+3Hx+G=0$. ... (1)

Here $f(x, y)\equiv x^3+3Hxy^2+Gy^3$.

Hence $$\frac{\partial f}{\partial x}=3(x^2+Hy^2), \qquad \frac{\partial f}{\partial y}=3(2Hxy+Gy^2),$$

$$\frac{\partial^2 f}{\partial x^2}=6x, \quad \frac{\partial^2 f}{\partial x \partial y}=6Hy, \quad \frac{\partial^2 f}{\partial y^2}=6(Hx+Gy).$$

The Hessian quadratic is, therefore, $x(Hx+Gy)=H^2y^2$, which gives
$$Hx^2+Gx-H^2=0 \qquad \ldots (2)$$
as the quadratic (6) of Art. 5·34.

When the roots of (2) are equal, $G^2+4H^3=0$,
which shows that two roots of (1) are also equal.

Also $$H^2(x^3+3Hx+G)\equiv(Hx^2+Gx-H^2)(Hx-G), \qquad \ldots (3)$$
when
$$G^2+4H^3=0.$$

This shows that the Hessian quadratic becomes a factor of the cubic.

5·37. When the roots of the cubic are all real, (7) Art. 5·34 shows that L, M cannot be real. Thus the method of the article fails entirely when the roots *are real and unequal.*

In this case, Cardan's method is also said to fail, for when G^2+4H^3 is $-ve.$, u^3 and v^3 are both complex, and there is no algebraical method of extracting the cube root of complex numbers. The problem can, however, be solved either by the Trigonometrical method explained later, or by the Trial Method of the next article.

5·38. It has been pointed out that if a cubic has all the three roots real, Cardan's Method fails to give the result in a neat form, for no general method exists for extracting the cube root of a complex number. Such equations can be solved, in several cases, by using the *Trial Method* in finding the cube roots of u^3 and v^3. (See Ex. 2 below). Similar statement holds true, even if the cubic has one real root, but the values of u^3 and v^3 are irrational.

Ex. 1. Solve the cubic $x^3+3x-14=0$ by Cardan's Method. (*Davison*)

Let $$x=u+v,$$
$$\therefore \quad x^3-3uvx-(u^3+v^3)=0$$

Identifying this with $x^3+3x-14=0$, we have
$$uv=-1, \; u^3+v^3=14.$$

Thus u^3 and v^3 are the roots of the equation

$$t^2-14t-1=0.$$

$$\therefore \quad u^3, v^3=7\pm5\sqrt{2}.$$

$\because$ $u^3, v^3=7\pm5\sqrt{2}$, u will be of the form $p+q\sqrt{2}$, and v of the form $p-q\sqrt{2}$.

$$\therefore \quad 7+5\sqrt{2}=p^3+2q^3\sqrt{2}+6q^2p+3pq^2\sqrt{2},$$
$$7-5\sqrt{2}=p^3-2q^3\sqrt{2}+6q^2p-3pq^2\sqrt{2}.$$

Hence
$$7=p^3+6q^2p=p(p^2+6q^2),$$
$$5=q(2q^2+3p^2).$$

The forms of u and v show that p and q are both positive; also p is a factor of 7 and q is a factor of 5.

Thus $p=1$, or 7 ; $q=1$, or 5.

Substitution shows that $p=1$, $q=1$ satisfy these equations.

Thus $u=1+\sqrt{2}$, $v=1-\sqrt{2}$.

Hence the roots are

$$u+v=2, \quad \omega u+\omega^2 v=-1+\sqrt{-6}.$$

The third root may obviously be written as $-1-\sqrt{-6}$.

Note.—This example shows that even if a cubic has only one real root, Cardan's Method does not always give the result in a neat form without trial at some stage.

Ex. 2. Solve the cubic $x^3-7x+6=0$ by Cardan's Method.

Let
$$x=u+v,$$
$$\therefore \quad x^3-3uvx-(u^3+v^3)=0.$$

Identifying with $x^3-7x+6=0$, we have

$$uv=\tfrac{7}{3}, \quad u^3+v^3=-6.$$

Hence u^3 and v^3 are the roots of

$$t^2+6t+\frac{343}{27}=0.$$

$$\therefore \quad u^3, v^3=-3\pm\frac{10i}{\sqrt{27}}=\frac{81\pm30\sqrt{-3}}{-27}.$$

Suppose $81+30\sqrt{-3}=(p+q\sqrt{-3})^3$

$$=p^3+3p^2q\sqrt{-3}-9pq^2-3q^3\sqrt{-3}.$$

$$81-30\sqrt{-3}=(p-q\sqrt{-3})^3$$

$$=p^3-3p^2q\sqrt{-3}-9pq^2+3q^3\sqrt{-3}.$$

$$\therefore \quad p(p^2-9q^2)=81, \quad q(p^2-q^2)=10.$$

These equations show that p is a factor of 81 and q of 10.

$\therefore$ q has the possible values ±1, ±2, ±5, ±10 ;
and p the values ±1, ±3, ±9, ±27, ±81.

Hence $q=2, p=-3.$

$$\therefore \quad u=\frac{-3+2\sqrt{-3}}{-3}, \quad v=\frac{-3-2\sqrt{-3}}{-3}.$$

Consequently $x=u+v=2,$

$$\omega u+\omega^2 v=-\tfrac{1}{3}\{-3(\omega+\omega^2)+2\sqrt{-3}(\omega-\omega^2)\}=1,$$

and

$$\omega^2 u+\omega v=-3.$$

NOTE 1. Having found one root, *e.g.*, 2, it will be convenient to divide the expression x^3-7x+6 by $x-2$, and then deduce the other two roots.

NOTE 2. *This example shows that some irreducible cases can be dealt with by Cardan's Method by using the method of trial at some stage. That the solution without the trial method is impossible, can be shown as follows :—*

To find p and q, let us solve the simultaneous equations

$$p(p^2-9q^2)=81, \qquad q(p^2-q^2)=10.$$

Put $p=\lambda q$, then the elimination of q gives the cubic

$$10\lambda^3-81\lambda^2-90\lambda+81=0,$$

which, by the transformation $10\lambda-27=-21z,$

becomes

$$z^3-7z+6=0,$$

and this is the same as the original equation.

That the result holds for the general case, can be proved by following the steps indicated above.

Examples V (*d*)

Solve the following equations :—

1. $x^3-26x+60=0.$
2. $x^3+x+10=0.$
3. $x^3-7x^2+36=0.$
4. $x^3-3x^2-10x+24=0.$

5·4. Descartes' solution of the biquadratic.

Let the equation be

$$a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0. \qquad \ldots(1)$$

If we put $z=a_0x+a_1$, the equation reduces to

$$z^4+6Hz^2+4Gz+a_0^2I-3H^2=0. \qquad \ldots(2)$$

Suppose that this biquadratic is resolved into two quadratic factors, $z^2+pz+q,\ z^2-pz+q'.$

Comparing the coefficients of the left-hand side of (2) and the product $(z^2+pz+q)(z^2-pz+q'),$ ...(3)

we get

$$6H=(q+q')-p^2, \qquad \ldots(4)$$

$$4G=p(q'-q), \qquad \ldots(5)$$

$$a_0^2I-3H^2=qq'. \qquad \ldots(6)$$

From (4) and (5), we get

$$q+q'=6H+p^2, \quad q'-q=\frac{4G}{p}.$$

Therefore $2q'=6H+p^2+\frac{4G}{p}$ and $2q=6H+p^2-\frac{4G}{p}$. ...(7)

Hence $(6H+p^2)^2-\frac{16G^2}{p^2}=4qq'=4(a_0^2I-3H^2)$,

i.e., $p^6+12Hp^4+4p^2(12H^2-a_0^2I)-16G^2=0$. ...(8)

This is a cubic in p^2 and is called *Descartes' Resolvent.*

Having found p from (8), q and q' can be obtained from (7). Thus the quadratic factors in (3) are completely determined and, consequently, the biquadratic can be solved by the help of two quadratic equations.

If we put $p^2+4H=t$, the equation reduces to

$$t^3-4a_0^2It+16a_0^3J=0. \quad ...(9)$$

$\because$ $G^2+4H^3=a_0^2(HI-a_0J)$,

If again, we put $t=-4a_0\phi$, the equation becomes the *reducing cubic*

$$4\phi^3-I\phi-J=0. \quad ...(10)$$

5·41. Relations between the roots of the biquadratic and Descartes' resolvent.

Let z_1, z_2, z_3, z_4, be the roots of the equation (2), and suppose z_1, z_2; z_3, z_4; are, respectively, the roots of $z^2+pz+q=0$ and $z^2-pz+q'=0$,

then $z_1+z_2=-(z_3+z_4)=-p$,

$\therefore$ $(z_1+z_2)^2=(z_3+z_4)^2=p^2$.

Hence the roots of Descartes' resolvent are

$$\left.\begin{aligned}&(z_1+z_2)^2,\ (z_2+z_3)^2,\ (z_3+z_1)^2;\\ \text{or}\quad &(z_3+z_4)^2,\ (z_1+z_4)^2,\ (z_2+z_4)^2.\end{aligned}\right\} ...(A)$$

The roots of Descartes' resolvent are connected with those of the reducing cubic by the relation

$$p^2+4H=-4a_0\phi.$$

Now if p_1^2, p_2^2, p_3^2 be the roots of Descartes' resolvent and ϕ_1, ϕ_2, ϕ_3 the corresponding roots of the reducing cubic,

$$-4a_0\phi_1=p_1^2+4H=(z_1+z_2)^2+4H,$$

and $$-4a_0\phi_2=p_2^2+4H=(z_2+z_3)^2+4H.$$

$\therefore$ $4a_0(\phi_2-\phi_1)=(z_1+z_2)^2-(z_2+z_3)^2$

$=(z_1-z_3)(z_1+z_3+2z_2)$

$=(z_1-z_3)(z_2-z_4)$ [$\because$ $z_1+z_2+z_3+z_4=0$.

$=a_0^2(\alpha-\gamma)(\beta-\delta)$. [$\because$ $a_0x+a_1=z$.

$\therefore$ $4(\phi_1-\phi_2)=a_0(\gamma-\alpha)(\beta-\delta)$

Similarly $4(\phi_2-\phi_3)=a_0(\alpha-\beta)(\gamma-\delta)$,

$4(\phi_3-\phi_1)=a_0(\beta-\gamma)(\alpha-\delta)$.

5·42. Ferrari's solution of the biquadratic.

Let the quartic be

$$a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0. \qquad \ldots(1)$$

Suppose $a_0(a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4)$

$$\equiv(a_0x^2+2px+r)(a_0x^2+2qx+s)$$

$\therefore$

$$p+q=2a_1, \qquad (i)$$

$$a_0(r+s)+4pq=6a_0a_2, \qquad (ii)$$

$$ps+qr=2a_0a_3, \qquad (iii)$$

$$rs=a_0a_4. \qquad (iv)$$

Also assume

$$r+s=2(a_2-2\phi), \qquad (v)$$

$\therefore$ by (ii and v)

$$pq=a_0a_2+a_0\phi.$$

Now $0=\begin{vmatrix} a_0 & a_0 & 0 \\ p & q & 0 \\ r & s & 0 \end{vmatrix} \begin{vmatrix} a_0 & a_0 & 0 \\ q & p & 0 \\ s & r & 0 \end{vmatrix}$

$$=\begin{vmatrix} 2a_0^2 & a_0(p+q) & a_0(r+s) \\ a_0(p+q) & 2pq & ps+qr \\ a_0(r+s) & ps+qr & 2rs \end{vmatrix}$$

$$=8a_0^3\begin{vmatrix} a_0 & a_1 & a_2-2\phi \\ a_1 & a_2+\phi & a_3 \\ a_2-2\phi & a_3 & a_4 \end{vmatrix},$$

which reduces to

$$4\phi_3-\mathrm{I}\phi-\mathrm{J}=0. \qquad \ldots(2)$$

Also from (iv) and (v), r and s are the roots of the quadratic

$$t^2-2(a_2-2\phi)+a_0a_4=0. \qquad \ldots(3)$$

If we now take any root of (2), r and s are determined from (3); then (i) and (iii) give the corresponding values of p and q. The biquadratic is then broken up into two quadratic factors, and the roots are found easily.

The equation (2) is called the *reducing cubic*. [Art. 5·4 (10)]

5·43. Relations between the roots of the biquadratic and its reducing cubic. Discriminant of the biquadratic.

Let ϕ_1, ϕ_2, ϕ_3, be the roots of the reducing cubic, and r_1, s_1, p_1, q_1 etc., the corresponding values of r, s, p and q.

Let α, β, γ, δ be the roots of the biquadratic, so that if α, β, are the roots of the quadratic

$$a_0x^2+2p_1x+r_1=0,$$

γ, δ, are the roots of the quadratic

$$a_0x^2+2q_1x+s_1=0.$$

Then $a_0\alpha\beta=r_1$, $a_0\gamma\delta=s_1$.

Hence $a_0(\alpha\beta+\gamma\delta)=2(a_2-2\phi_1)$. [from (3), above.

Similarly $a_0(\beta\gamma+\alpha\delta)=2(a_2-2\phi_2)$.

and $a_0(\gamma\alpha+\beta\delta)=2(a_2-2\phi_3)$.

Thus
$$\left.\begin{aligned}4(\phi_2-\phi_3)&=a_0(\alpha-\beta)(\gamma-\delta),\\ 4(\phi_3-\phi_1)&=a_0(\beta-\gamma)(\alpha-\delta),\\ 4(\phi_1-\phi_2)&=a_0(\gamma-\alpha)(\beta-\delta).\end{aligned}\right\}\ (A)$$

Also $a_0^6(\alpha-\beta)^2(\gamma-\delta)^2(\beta-\gamma)^2(\alpha-\delta)^2(\gamma-\alpha)^2(\beta-\delta)^2$
$$=16^3(\phi_2-\phi_3)^2(\phi_3-\phi_1)^2(\phi_1-\phi_2)^2$$
$$=256(\mathrm{I}^3-27\mathrm{J}^2),$$

which is the *discriminant of the biquadratic.*

5·44. Geometrical significance of the algebraic solution of the biquadratic.

Let the biquadratic be

$$a_0t^4+4a_1t^3+6a_2t^2+4a_3t+a_4=0. \qquad \ldots(1)$$

Consider the conics

$$\mathrm{C}\equiv a_0x^2+2a_1xy+a_2y^2+2a_2x+2a_3y+a_4=0, \qquad \ldots(2)$$

and $y^2=4x$. The latter conic, which is a parabola, is in the parametric form

$$x=t^2,\ y=2t \qquad \ldots \quad \ldots \quad \ldots(3)$$

Now the parameters of the four points of intersection of (2) and (3) are the roots of the equation (1).

The equation

$$\mathrm{F}\equiv\mathrm{C}+\phi.(y^2-4x)=0,$$

or $$a_0x^2+2a_1xy+(a_2+\phi)y^2+2(a_2-2\phi)x+2a_3y+a_4=0, \qquad \ldots(4)$$

gives the pencil of conics whose base points are the intersections of $\mathrm{C}=0$ and $y^2=4x$.

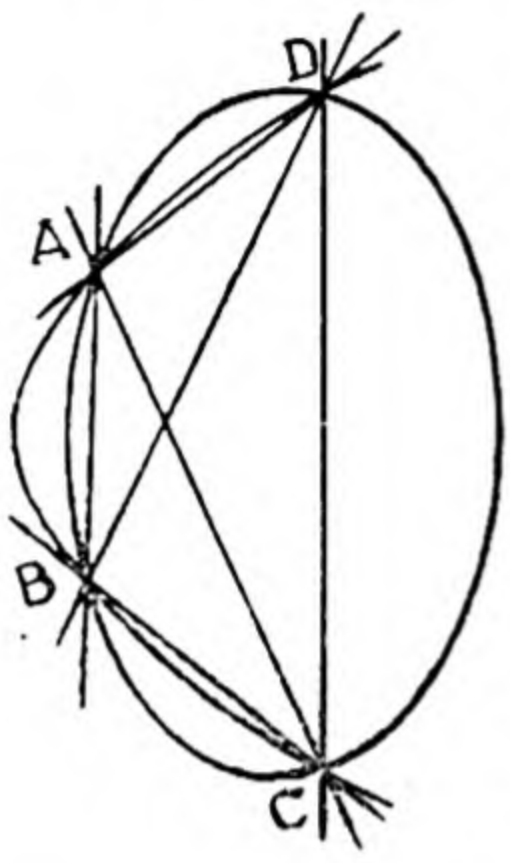
The system (4) will degenerate into a pair of straight lines for three values of ϕ which are the roots of the equation

$$\begin{vmatrix} a_0 & a_1 & a_2-2\phi \\ a_1 & a_2+\phi & a_3 \\ a_2-2\phi & a_3 & a_4 \end{vmatrix}=0,$$

which reduces to

$$4\phi^3-\mathrm{I}\phi-\mathrm{J}=0.$$

Comparing this equation with the equations in arts. 5·4 and 5·42, we see that this is the reducing cubic. For any such value of ϕ, F will break into two linear expressions in x, y, and when $x=t^2$, $y=2t$ are substituted in F, the term y^2-4x disappears, and F becomes C which takes the form of the quartic (1) and the two linear expressions in x, y become the quadratic factors of (1).



5·45. Graphical solution of biquadratic.

The preceding art. suggests the method. Two conics, in general position, intersect in four points, The attempt should therefore be to construct two conics, the abscissae (or ordinates) of whose points of intersection are the roots of the given equation. The ingenuity lies in constructing the curves which can be easily drawn. Among the second degree curves, such curves are a circle and a parabola (given by the standard equation).

Let the equation be

$$x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0,$$

or $$x^2+(x^2+2a_1x)^2+(6a_2-4a_1^2-1)x^2+4a_3x+a_4=0.$$

Now take $y=x^2+2a_1x$ (i) as one of the curves; the preceding equation then becomes

$$x^2+y^2+(6a_2-4a_1^2-1)(y-2a_1x)+4a_3x+a_4=0,$$

i.e., $$x^2+y^2+(6a_2-4a_1^2-1)y+2(2a_3-6a_1a_2+4a_1^3+a_1)x+a_4=0. \quad \ldots(ii)$$

The solution then depends on the parabola (i) and the circle (ii).

5·46. Nature of the roots of the biquadratic.*

Let $a_0x_4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$ be the equation, whose roots are x_1, x_2, x_3, x_4; and let z_1, z_2, z_3, z_4, be the roots of the reduced equation $z^4+6Hz^2+4Gz+a_0^2I-3H^2=0$.

Then $$(z_1-z_2)^2(z_1-z_3)^2(z_1-z_4)^2(z_2-z_3)^2(z_2-z_4)^2(z_3-z_4)^2$$
$$=a_0^{12}(x_1-x_2)^2(x_1-x_3)^2(x_1-x_4)^2(x_2-x_3)^2(x_2-x_4)^2(x_3-x_4)^2$$
$$=256\,a_0^6(I^3-27\,J^2)=256\,a_0^6\triangle. \quad \ldots(i)$$

Suppose that $z_1=\alpha+\sqrt{\beta^2}$, $z_2=\alpha-\sqrt{\beta^2}$,

$z_3=-\alpha+\sqrt{\gamma^2}$, $z_4=-\alpha-\sqrt{\gamma^2}$,

then $$\Sigma z_1z_2=-(2\alpha^2+\beta^2+\gamma^2)=6H, \quad \ldots(ii)$$

*Hukam Chand: Journal of the Indian Mathematical Society, Vol. XVIII, No. 12, page 292.

$$\Sigma z_1z_2z_3 = \quad 2\alpha\ (\beta^2-\gamma^2) = -4G, \qquad \ldots(iii)$$
$$z_1z_2z_3z_4 = (\alpha^2-\beta^2)\,(\alpha^2-\gamma^2) = a_0{}^2I - 3H^2, \qquad \ldots(iv)$$

and the relation (i) becomes

$$16\ \beta^2\gamma^2\,[(4\alpha^2-\beta^2-\gamma^2)^2-4\beta^2\gamma^2]^2 = 256\ a_0{}^6\,\Delta. \qquad \ldots(v)$$

From (iv) subtract $\frac{1}{4}$th the square of (ii),

then
$$-2\alpha^2(\beta^2+\gamma^2) - \frac{(\beta^2-\gamma^2)^2}{4} = a_0{}^2I - 12H^2. \qquad \ldots(vi)$$

Case I. When all the roots are complex, $\beta^2<0$, $\gamma^2<0$, $\alpha^2\geqslant 0$, then relation (v) shows that $\Delta > 0$.

Case II. When all the roots are real and distinct, $\beta^2>0$, $\gamma^2>0$, $\alpha^2\geqslant 0$; then relation (v) shows that $\Delta>0$, and (ii) and (vi) show that $H<0$, $a_0{}^2I-12H^2<0$.

Case III. When two roots are real and two complex, $\alpha^2>0$, $\beta^2>0$, $\gamma^2<0$; then relation (v) shows that $\Delta<0$.

Case IV. (a) Two roots equal; then $\gamma^2=0$, $\beta^2\neq 0$; hence relation (v) gives $\Delta=0$.

(b) Three roots equal; then $\alpha^2>0$, $\gamma^2=0$; $\therefore\ z_3=z_4=-\alpha$, and $\beta^2=4\alpha^2$. From (ii) $\alpha^2=-H$, and so from (iv), $I=0$, and since (v) gives $I^3-27\ J^2=0$, $J=0$.

Consequently the conditions are $I=0$, $J=0$.

(c) When all the roots are equal $\alpha=0$, $\beta=0$, $\gamma=0$, hence $H=0$, $G=0$; from (iv), $I=0$ and from (v) $J=0$.

(d) When there are two distinct pairs of equal roots, either all the roots are real or they are all complex. In the first case, $\beta=0=\gamma$, and $\alpha\neq 0$. In the latter case, $\alpha=0$ and $\beta=\gamma\neq 0$.

In both the cases, (ii) gives $G=0$, and (vi) gives

$$a_0{}^2I = 12H^2.$$

The conditions in (b), (c) and (d) imply the relation $\Delta=0$.

EXAMPLES V (e)

1. Solve $x^4-8x^3+9x^2+8x-10=0$.

This equation can be written as

$$(x^2-4x)^2 = 7x^2-8x+10,$$

or
$$(x^2-4x+\lambda)^2 = (7+2\lambda)x^2-8x(1+\lambda)+(10+\lambda^2).$$

Choose λ so that the right-hand side be a perfect square. This requires that

$$16(1+\lambda)^2 = (\lambda^2+10)(2\lambda+7),$$

or
$$2\lambda^3-9\lambda^2-12\lambda-54=0.$$

We may multiply the roots of this equation by 2; putting $2\lambda=z$, the equation becomes

$$z^3-9z^2-24z+216=0,$$

which is satisfied by $z=9$, $\therefore\ \lambda=\frac{9}{2}$.

The biquadratic then becomes

$$(x^2-4x+\tfrac{9}{2})^2=16x^2-44x+\tfrac{161}{4}$$
$$=(4x-\tfrac{11}{2})^2,$$

or $$x^2-4x+\tfrac{9}{2}=\pm(4x-\tfrac{11}{2}).$$

The two quadratics are

$$x^2-8x+10=0, \quad x^2=1.$$

The roots are therefore ± 1, $4\pm\sqrt{6}$.

2. Solve the equation $x^4-8x^2-24x+7=0$. (*P. U. 1931*)

Let $x^4-8x^2-24x+7\equiv(x^2+px+q)(x^2-px+q')$.

Then
$$q'+q-p^2=-8, \qquad (i)$$
$$p(q'-q)=-24. \qquad (ii)$$
$$qq'=7. \qquad (iii)$$

$$\therefore\ 28=4qq'=(q+q')^2-(q-q')^2=(p^2-8)^2-\frac{576}{p^2},$$

or $t^3-16t^2+36t-576=0$, where $p^2=t$.

Evidently $p^2=t=16$. Taking $p=4$,

from (i) and (ii) $q'=1$, $q=7$. Thus the two quadratics are

$$x^2+4x+7=0 \text{ and } x^2-4x+1=0.$$

$$\therefore\ x=-2\pm\sqrt{-3}, \quad 2\pm\sqrt{3}.$$

3. Solve graphically the equation $x^4+2x^3-7x^2-8x+12=0$.

The equation can be written as

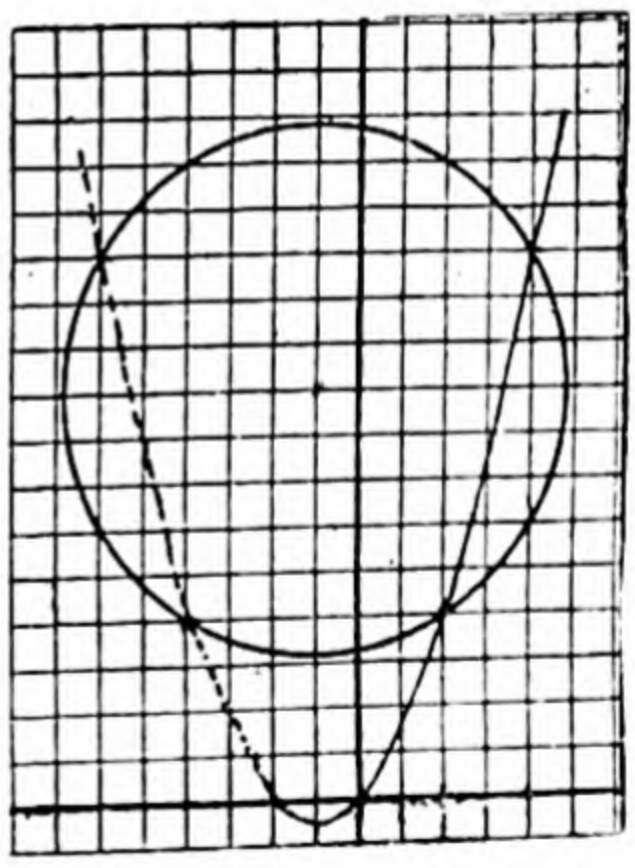
$$(x^2+x)^2+x^2-9x^2-8x+12=0.$$

Put $$y=x^2+x, \qquad (i)$$

$$\therefore\ x^2+y^2-9(y-x)-8x+12=0$$

or $$x^2+y^2+x-9y+12=0 \qquad (ii)$$

The roots of the equation are the abscissae of the intersections of (i) and (ii). The circle has its centre at $(-\frac{1}{2}, \frac{9}{2})$ and passes through $(-3, 3)$. Writing the equation of the parabola as $y+\frac{1}{4}=(x+\frac{1}{2})^2$, the vertex of the parabola is $(-\frac{1}{2}, -\frac{1}{4})$. (Scale 1 unit $=2$ side of a square). The abscissae of the points of intersection are $+1$, 2. -2, -3.



4. Solve $t^4+2t^3-14t^2+8=0$.

Consider the parabola $y^2=4x$,

$$\therefore\ x=t^2,\ y=2t.$$

Writing the biquadratic as

$$t^4+2t^3-4t^2-10t^2+8=0,$$

it is seen that the biquadratic gives the parameters of the points of intersection of the above parabola and the conic

$$C\equiv x^2+xy-y^2-10x+8=0.$$

The conic $S \equiv x^2+xy-y^2-10x+8+\phi(y^2-4x)=0$,
or $x^2+xy+y^2(\phi-1)-2x(5+2\phi)+8=0$
will degenerate if $8(\phi-1)-(\phi-1)(5+2\phi)^2-8.\frac{1}{4}=0$,
i.e., $4\phi^3+16\phi^2-3\phi-15=0$.
$\phi=-1$.

Thus S becomes
$x^2+xy-2y^2-6x+8=0$, or $(x+2y-4)(x-y-2)=0$.
The two quadratics are therefore
$t^2+4t-4=0$, $t^2-2t-2=0$,
which give $t=-2\pm2\sqrt{2}$, $1\pm\sqrt{3}$.

5. Solve the following equations :—

(i) $x^4-2x^2+8x-3=0$. (*P. U. 1943*)
(ii) $x^4-10x^2-20x-16=0$. (*P. U. 1941*)
(iii) $x^4-5x^2-6x-5=0$. (*Queen's Coll., Ox., P. U. 1935*)
(iv) $x^4+12x-5=0$. (*Trinity*)
(v) $x^4-2x^3+4x^2+6x-21=0$. (*Queen's*)
(vi) $x^4-10x^3+35x^2-50x+24=0$. (*P. U. 1939*)
(vii) $x^4+3x^3-2x^2-7x+3=0$. (*Radford*)
(viii) $x^4-10x^3+26x^2-10x+1=0$. (*P. U. 1918*)
(ix) $2x^4+6x^3-3x^2+2=0$. (*P. U. 1933*)

6. Solve the equation $x^4+12x+3=0$ by resolving it into its quadratic factors, (*Queen's*)

7. Show that values of A, B; a, b; can be found which make $A(x+a)^4+B(x+b)^4=x^4-24x^2-80x-84$ for all values of x; hence, or otherwise, find all the roots of the equation

$$x^4-24x^2-80x-84=0.$$

8. If the equation $x^4+6ax^2+4bx+c=0$ can be written in the form $m(x-n)^4-n(x-m)^4=0$, show that $a^3+b^2=ac$, and that m and n are the roots of the equation $at^2+bt-a^2=0$.

Solve the equation $x^4+12x^2+8x+6=0$.

9. Prove that J vanishes for the equation

$$m(x-n)^4-n(x-m)^4=0.$$

10. Show that the condition, that the biquadratic may be capable of being put under the form
$l(x^2+2px+q)^2+m(x^2+2px+q)+n=0$, is that $G=0$.

[**Hint.** The same transformation removes the second and the fourth coefficients.]

11. If α, β, γ, δ be the roots of the equation

$$ax^4+4bx^3+6cx^2+4dx+e=0 \qquad (1)$$

form the equation whose roots are

(i) $\alpha\beta+\gamma\delta$, $\alpha\gamma+\beta\delta$, $\alpha\delta+\beta\gamma$;
(ii) $(\alpha+\beta)(\gamma+\delta)$, $(\alpha+\gamma)(\beta+\delta)$, $(\alpha+\delta)(\beta+\gamma)$;
(iii) $(\alpha+\beta-\gamma-\delta)^2$, $(\alpha+\gamma-\beta-\delta)^2$, $(\alpha+\delta-\beta-\gamma)^2$.

Suppose that $\alpha, \beta\,;\ \gamma, \delta\,;$ are the roots of the quadratics

$$x^2-px+q=0 \quad \text{and} \quad x^2-p'x+q'=0.$$

The given biquadratic is, therefore, identical with

$$(x^2-px+q)(x^2-p'x+q')=0. \tag{2}$$

Identifying the given biquadratic (1) with (2),

$$-(p+p')=\frac{4b}{a},\quad q+q'+pp'=\frac{6c}{a},$$

$$-(pq'+p'q)=\frac{4d}{a},\quad qq'=\frac{e}{a}.$$

Now
$$0=\begin{vmatrix} 1 & 1 & 0 \\ -p & -p' & 0 \\ q & q' & 0 \end{vmatrix}\begin{vmatrix} 1 & 1 & 0 \\ -p' & -p & 0 \\ q' & q & 0 \end{vmatrix}$$

$$=\begin{vmatrix} 2 & -(p+p') & q+q' \\ -(p+p') & 2pp' & -(pq'+p'q) \\ q+q' & -(pq'+p'q) & 2qq' \end{vmatrix}. \tag{3}$$

(i) Let $t=\alpha\beta+\gamma\delta=q+q'$,

$$\therefore\ pp'=\frac{6c}{a}-t.$$

Substituting in (3), we obtain the equation

$$\begin{vmatrix} 2 & \frac{4b}{a} & t \\ \frac{4b}{a} & 2\left(\frac{6c}{a}-t\right) & \frac{4d}{a} \\ t & \frac{4d}{a} & \frac{2e}{a} \end{vmatrix}=0, \text{ or } \begin{vmatrix} 2a & 2b & at \\ 2b & \frac{1}{2}(6c-at) & 2d \\ at & 2d & 2e \end{vmatrix}=0.$$

(ii) Let $(\alpha+\beta)(\gamma+\delta)=y$, so that $y=pp'$.

$$\therefore\ q+q'=\frac{6c}{a}-y.$$

Substituting in (3), we get the equation

$$\begin{vmatrix} 2a & 2b & 6c-ay \\ 2b & ay/2 & 2d \\ 6c-ay & 2d & 2e \end{vmatrix}=0,$$

whose roots are $(\alpha+\beta)(\gamma+\delta)$, and two similar expressions.

(iii) Let $z=(\alpha+\beta-\gamma-\delta)^2=(p'-p)^2=(p+p')^2-4pp'$

$$=\frac{16b^2}{a^2}-4pp'.$$

$$\therefore\ pp'=\tfrac{1}{4}\left(\frac{16b^2}{a^2}-z\right),$$

also
$$q+q'=\frac{6c}{a}-pp'$$

$$=\frac{6c}{a}-\tfrac{1}{4}\left(\frac{16b^2}{a^2}-z\right)$$

$$=\frac{6ac-4b^2}{a^2}+\tfrac{1}{4}z.$$

Substituting in (3) again, we get the equation

$$\begin{vmatrix} 2 & \frac{4b}{a} & \frac{6ac-4b^2}{a^2}+\frac{1}{4}z \\ \frac{-4b}{a} & \frac{1}{2}\left(\frac{16b^2}{a^2}-z\right) & \frac{4d}{a} \\ \frac{6ac-4b^2}{a^2}+\frac{1}{4}z & \frac{4d}{a} & \frac{2e}{a} \end{vmatrix}=0.$$

CHAPTER VI

ELEMENTARY INEQUALITIES

The present chapter deals with elementary inequalities, some of which are of fundamental importance in various branches of the subject.

6·1. If $a>b$, *then*

(i) $a+x>b+x$, $a-x>b-x$ ...(·101)

(ii) $-a<-b$ ...(·102)

(iii) $ma>mb$, $-ma<-mb$, *when* $m>0$. ...(·103)

Proof. (i) $\because$ $(a+x)-(b+x)=a-b>0$,

$\therefore$ $a+x>b+x$,

The second result follows similarly.

(ii) Take $x=a+b$ in the second part of (i)

(iii) $\because$ $ma-mb=m(a-b)>0$

if $m>0$ for $a>b$,

$\therefore$ $ma>mb$.

The second result follows from the fact that

$$-m(a-b)<0, \text{ if } m>0.$$

The student will note that if an inequality be multiplied or divided by a negative number, the sign of the inequality is reversed.

Thus while an inequality continues to hold after each side has been increased or decreased by any quantity, positive or negative, it holds after multiplication or division only when the multiplying or the dividing factor is positive. In case of its being negative, the inequality is reversed.

Ex. 1. If a, b, c be any real quantities, not all equal, prove that

$$a^2+b^2+c^2>ab+bc+ca.$$

What happens when they are all equal ?

Ex. 2. If a and b be any two positive unequal numbers, prove that

$$\frac{a+b}{2}>\sqrt{ab}.$$

6·11. If $a_1>a_2$, $a_2>a_3 \ldots\ldots a_{n-1}>a_n$, then $a_1>a_n$ ...(·111)

Now $a_1-a_n=(a_1-a_2)+(a_2-a_3)+\ldots\ldots+(a_{n-1}-a_n)>0$, for each term on the right is positive.

$\therefore$ $a_1>a_n$.

6·12. If $a_1, b_1; a_2, b_2; \ldots a_n, b_n$ are all positive, and

$$a_1>b_1,\ a_2>b_2, \ldots\ldots a_n>b_n, \text{ then}$$

$$a_1+a_2+\ldots\ldots+a_n>b_1+b_2+\ldots\ldots+b_n, \qquad \ldots(\cdot 121)$$

$$a_1a_2\ldots\ldots a_n>b_1b_2\ldots\ldots b_n. \qquad \ldots(\cdot 122)$$

Proof.
$$\begin{aligned} a_1+a_2+\ldots+a_n &>b_1+a_2+\ldots+a_n, \\ &>b_1+b_2+a_2+\ldots+a_n \\ &>b_1+b_2+\ldots\ldots\ldots+b_n \\ \therefore\ a_1+a_2+\ldots+a_n &>b_1+b_2+\ldots\ldots\ldots+b_n. \end{aligned}$$

To prove (·122), we note that

$$\begin{aligned} a_1a_2a_3\ldots\ldots a_n &>b_1a_2a_3\ldots\ldots a_n && \because\ a_2a_3\ldots a_n>0 \\ &>b_1b_2a_3\ldots\ldots a_n && \because\ a_3a_4\ldots a_n>0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &>b_1b_2\ldots\ldots b_n. \end{aligned}$$

6·13. If $a>b>0$, and n is a positive rational, then

$$a^n>b^n. \qquad \ldots(\cdot 13)$$

Proof. *Case* (*i*). *When n is a positive integer.*

The result follows from (·122) when

$$a_1=a_2=\ldots\ldots=a_n=a,\ b_1=b_2=\ldots\ldots b_n=b.$$

Case (*ii*). Let n be a positive rational,

then $$n=p/q,\ p>0,\ q>0.$$

In the first place $$a^{\frac{1}{q}}>b^{\frac{1}{q}}.$$

For, if not, let $$a^{\frac{1}{q}}\leqslant b^{\frac{1}{q}},$$

then $$\left(a^{\frac{1}{q}}\right)^q\leqslant\left(b^{\frac{1}{q}}\right)^q, \qquad \text{[by case (i)]}$$

i.e., $$a\leqslant b,$$

which contradicts the condition $a>b$;

$$\therefore\ a^{\frac{1}{q}}>b^{\frac{1}{q}}$$

Hence $$\left(a^{\frac{1}{q}}\right)^p>\left(b^{\frac{1}{q}}\right)^p, \qquad \text{[by case (i)]}$$

or $$a^n>b^n \qquad \because\ n=p/q$$

We have incidentally proved that

$$\text{if } a>b,\ a^{\frac{1}{n}}>b^{\frac{1}{n}} \text{ when } n>0.$$

In this case it is assumed that the real positive value of the nth root is taken. Similarly the real positive value of the qth root is taken in the proof of Case (ii).

6·131. If $a>b$, and n is a negative rational, then

$$a^n<b^n \qquad \ldots(\cdot 131)$$

Let $n=-m$, so that $m>0$; by (6·13) $a^m>b^m$. Dividing by a^mb^m, which is positive, $b^{-m}>a^{-m}$, *i.e.*, $b^n>a^n$, or $a^n<b^n$.

6·2. Weirstrass' Inequalities. *If $a_1, a_2, \ldots a_n$ are positive numbers less than unity, and $a_1+a_2+\ldots+a_n=s_n$, then*

$$1-s_n<(1-a_1)(1-a_2)\ldots\ldots(1-a_n)<\frac{1}{1+s_n}, \qquad (\cdot 21)$$

$$1+s_n<(1+a_1)(1+a_2)\ldots\ldots\ldots(1+a_n)<\frac{1}{1-s_n}, \qquad (\cdot 22)$$

where, in the last inequality, it is supposed that $s_n<1$.

Proof: $(1-a_1)(1-a_2)=1-(a_1+a_2)+a_1a_2$

$>1-(a_1+a_2)$, *(ii)*

$(1-a_1)(1-a_2)(1-a_3)>\{1-(a_1+a_2)\}(1-a_3)$

$>1-(a_1+a_2+a_3)$. [by *(i)*]

Continuing this process, we prove that

$$(1-a_1)(1-a_2)\ldots\ldots\ldots(1-a_n)>1-s_n.$$

Again, $(1+a_1)(1+a_2)=1+(a_1+a_2)+a_1a_2$

$>1+(a_1+a_2)$,

$(1+a_1)(1+a_2)(1+a_3)>\{1+(a_1+a_2)\}(1+a_3)$

$>1+(a_1+a_2+a_3)$.

Proceeding in this way, we get

$$(1+a_1)(1+a_2)\ldots\ldots(1+a_n)>1+s_n.$$

Now $1-a_1^2<1$, $\therefore\ 1-a_1<\frac{1}{1+a_1}$ and $1+a_1<\frac{1}{1-a_1}$,

hence

$$(1-a_1)(1-a_2)\ldots\ldots(1-a_n)<\frac{1}{(1+a_1)(1+a_2)\ldots(1+a_n)}<\frac{1}{1+s_n}.$$

This proves the first part of the inequality.

Similarly, $1+s_n<(1+a_1)(1+a_2)\ldots\ldots(1+a_n)$

$$<\frac{1}{(1-a_1)(1-a_2)\ldots(1-a_n)}<\frac{1}{1-s_n}.$$

This establishes the second part of the inequality.

Ex. 1. If n is a positive integer, and $0<a<1$, prove without assuming the above formulæ that

(i) $1-na<(1-a)^n<\frac{1}{1+na}$,

(ii) $1+na<(1+a)^n<\frac{1}{1-na}$.

In (ii) $na<1$.

Ex. 2. If n is a positive integer, and $1+x>0$ (excluding $x=0$), prove that $(1+x)^n>1+nx$.

6·3. The theorem of the arithmetic, geometric, and harmonic means.

Let $a_1, a_2, \ldots\ldots, a_n$ be n positive numbers ; put

$$A \equiv \frac{1}{n}(a_1+a_2+\ldots\ldots+a_n),$$

$$G \equiv (a_1 a_2 \ldots\ldots a_n)^{\frac{1}{n}},$$

$$H = \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}};$$

then A is called the *arithmetic mean*, G *the geometric mean*, and H the *harmonic mean* of $a_1, a_2, \ldots\ldots, a_n$.

To prove that $A>G>H$, *unless all the a's are equal.*

We shall first prove that $A>G$.

Obviously, $a_1 a_2 = \left(\frac{a_1+a_2}{2}\right)^2 - \left(\frac{a_1-a_2}{2}\right)^2 \leqslant \left(\frac{a_1+a_2}{2}\right)^2$.

It follows that

$$(a_1 a_2)(a_3 a_4) < \left(\frac{a_1+a_2}{2}\right)^2 \left(\frac{a_3+a_4}{2}\right)^2 < \left(\frac{a_1+a_2+a_3+a_4}{4}\right)^4;$$

there will be a sign of equality if $a_1=a_2=a_3=a_4$.

Repeating the argument m times, we have

$$a_1 a_2 \ldots\ldots a_n \leqslant \left(\frac{a_1+a_2+\ldots\ldots+a_n}{n}\right)^n, \text{ when } n=2^m,$$

the sign of equality holding only when all the a's are equal.

Next, let n be not of the form 2^m, and suppose m is the least integer such that $2^m>n$. Let $2^m=n+k$ and let the k numbers from a_{n+1} to a_{2^m} be all equal to $\frac{a_1+a_2+\cdots+a_n}{n}$, *i.e.* A ; then by the above theorem,

$$a_1 a_2 \ldots a_n . a_{n+1} \ldots\ldots a_{2^m} \leqslant \left(\frac{a_1+a_2+\ldots\ldots+a_n+\overline{a_{n+1}+\ldots a_{2^m}}}{2^m}\right)^{2^m},$$

or

$$a_1 a_2 \ldots a_n A^k \leqslant \left(\frac{a_1+a_2+\ldots\ldots+a_n+kA}{2^m}\right)^{2^m}$$

$$= \left[\frac{nA+kA}{2^m}\right]^{2^m} = A^{2^m}.$$

$$\therefore \quad a_1 a_2 \ldots\ldots a_n \leqslant A^{2m-k} < A^n. \qquad \ldots(6)$$

$\therefore$ $A>G$, when all the a's are not equal.

Second Proof. In view of the importance of the theorem we add another proof depending on different considerations.

Let a_1 be the greatest and a_2 the least a in the set of numbers, then

$$na_1 > a_1+a_2+\ldots\ldots a_n > na_2, \quad \therefore \quad a_1 > A > a_2.$$

Consider the set

$$A,\ (a_1+a_2-A),\ a_3,\ldots\ldots a_n.$$

The arithmetic mean of this set is A; and the geometric mean is

$$[A(a_1+a_2-A)a_3\ldots\ldots a_n]^{\frac{1}{n}},$$

which is $$> (a_1 a_2 a_3\ldots\ldots a_n)^{\frac{1}{n}},$$

if $$A(a_1+a_2-A) > a_1 a_2,$$

or if $$0 > (A-a_1)(A-a_2),$$

which is true, since the right-hand side is negative, for $a_1 > A > a_2$. Thus the effect of the above change is to keep the value of A intact, and to increase the value of the geometric mean.

Repeating the above argument, we replace every number by A, and at the most n repetitions will be necessary. Finally, we have a set of n numbers each equal to A, of which the geometric mean is greater than G. But the geometric mean of a set of n numbers, each equal to A, is A. Hence

$$A > G. \qquad \ldots(6)$$

Proof of the second part of the theorem. By the above theorem

$$\frac{1}{n}\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right) > \left(\frac{1}{a_1}\cdot\frac{1}{a_2}\cdots\frac{1}{a_n}\right)^{\frac{1}{n}},$$

i.e., $$\frac{1}{H} > \frac{1}{G}, \text{ or } G > H.$$

6·31. *If $a_1, a_2, \ldots a_n$ be n positive numbers and $p_1, p_2 \ldots p_n$ another set of rational numbers, then*

$$\frac{p_1a_1+p_2a_2+\cdots+p_na_n}{p_1+p_2+\cdots+p_n} \geqslant \left(a_1^{p_1} a_2^{p_2} \ldots\ldots a_n^{p_n}\right)^{\frac{1}{p_1+p_2+\ldots\ldots p_n}} \qquad (\cdot 311)$$

As we are concerned with the ratios of $p_1 : p_2 \ldots\ldots$, and not with their actual values, we may replace $p_1, p_2, \ldots$ by integers $q_1, q_2, \ldots$ proportional to them, *i.e.*,

$$\frac{p_1}{q_1} = \frac{p_2}{q_2} = \ldots\ldots = \frac{p_n}{q_n} = k.$$

Making the substitution, the above inequality becomes

$$\frac{q_1a_1+q_2a_2+\cdots+q_na_n}{q_1+q_2+\cdots+q_n} \geqslant \left(a_1^{q_1}a_2^{q_2}\ldots a^{n^{q_n}}\right)^{\frac{1}{q_1q_2\cdots q_n}}.$$

This is only a particular case of Article 6·3 when q_1 numbers are equal to a_1, q_2 equal to a_2, and so on.

6·4. *If a_1, a_2 are positive numbers (including zero), and p, q, are positive integers, then*

$$\frac{a_1^{p+q}+a_2^{p+q}}{2} \geqslant \left(\frac{a_1^p+a_2^p}{2}\right)\left(\frac{a_1^q+a_2^q}{2}\right), \qquad (\cdot 401)$$

$$\frac{a_1^{p+q+r+\cdots}+a_2^{p+q+r+\cdots}}{2} \geqslant \left(\frac{a_1^p+a_2^p}{2}\right)\left(\frac{a_1^q+a_2^q}{2}\right)\left(\frac{a_1^r+a_2^r}{2}\right)\ldots (\cdot 402)$$

$$\frac{a_1^p+a_2^p}{2} \geqslant \left(\frac{a_1+a_2}{2}\right)^p. \qquad (\cdot 403)$$

Proof. Since $a_1^p-a_2^p$, $a_1^q-a_2^q$ have the same sign, or each of them is equal to zero,

$$(a_1^p-a_2^p)(a_1^q-a_2^q) \geqslant 0,$$

or $$a_1^{p+q}+a_2^{p+q} \geqslant a_1^pa_2^q+a_1^qa_2^p,$$

$$\therefore\ 2(a_1^{p+q}+a_2^{p+q}) \geqslant a_1^{p+q}+a_2^{p+q}+a_1^pa_2^q+a_1^qa_2^p,$$

i.e., $$\geqslant (a_1^p+a_2^p)(a_1^q+a_2^q).$$

Hence $$\frac{a_1^{p+q}+a_2^{p+q}}{2} \geqslant \frac{a_1^p+a_2^p}{2}\cdot\frac{a_1^q+a_2^q}{2}.$$

Again $$\frac{a_1^{p+q+r}+a_2^{p+q+r}}{2} \geqslant \frac{a_1^{p+q}+a_2^{p+q}}{2}\cdot\frac{a_1^r+a_2^r}{2}$$

$$\geqslant \frac{a_1^p+a_2^p}{2}\cdot\frac{a_1^q+a_2^q}{2}\cdot\frac{a_1^r+a_2^r}{2}.$$

By repeated application of this formula, we get the formula (·402).

Putting $p=q=r=\ldots=1$, in (·402), we obtain (·403).

6·41. Generalization to n numbers. *If a_1, a_2...denote n positive numbers including zero, and p, q,...positive integers,*

$$\frac{a_1^{p+q}+a_2^{p+q}+\ldots+a_n^{p+q}}{n} \geqslant \frac{a_1^p+a_2^p+\ldots+a_n^p}{n}\cdot\frac{a_1^q+a_2^q+\ldots+a_n^q}{n}$$

Proof. As $a_1^p-a_2^p$, $a_1^q-a_2^q$ have the same sign,

$$(a_1^p-a_2^p)(a_1^q-a_2^q) \geqslant 0,$$

or $$a_1^{p+q}+a_2^{p+q} \geqslant a_1^pa_2^q+a_1^qa_2^p.$$

We write down the ${}^nC_2$ such inequalities obtained by taking all the combinations of the numbers a_1, a_2,.... Adding them, we have

$$(n-1)\Sigma a_1^{p+q} \geqslant \Sigma a_1^p a_2^q.$$

But $$\Sigma a_1^p a_2^q = \Sigma a_1^p \Sigma a_1^q - \Sigma a_1^{p+q}.$$

$$\therefore\ n\Sigma a_1^{p+q} \geqslant \Sigma a_1^p \Sigma a_1^q,$$

or
$$\frac{1}{n}\Sigma a_1^{p+q} \geqslant \left(\frac{1}{n}\Sigma a_1^p\right)\left(\frac{1}{n}\Sigma a_1^q\right). \qquad (\cdot 411)$$

We may generalize this further, and have

$$\frac{1}{n}\Sigma a_1^{p+q+r} \geqslant \left(\frac{1}{n}\Sigma a_1^{p+q}\right)\left(\frac{1}{n}\Sigma a_1^r\right)$$

$$\geqslant \left(\frac{1}{n}\Sigma a_1^p\right)\left(\frac{1}{n}\Sigma a_1^q\right)\left(\frac{1}{n}\Sigma a_1^r\right).$$

Thus

$$\frac{1}{n}\Sigma a_1^{p+q+r+\cdots} \geqslant \left(\frac{1}{n}\Sigma a_1^p\right)\left(\frac{1}{n}\Sigma a_1^q\right)\left(\frac{1}{n}\Sigma a_1^r\right). \quad .. \qquad (\cdot 412)$$

In particular, putting $p=q=r\ldots=1$, *in* (·412),

$$\frac{1}{n}\Sigma a_1^p \geqslant \left(\frac{1}{n}\Sigma a_1\right)^p \qquad (\cdot 413)$$

The sign of the inequality in (·411) will change if p and q are of opposite signs. An interesting particular case of (·411) is when $q=-p$, then

$$\left(\frac{1}{n}\Sigma a_1^p\right)\left(\frac{1}{n}\Sigma a_1^{-p}\right) \geqslant 1,$$

or
$$\Sigma a_1^p \Sigma a_1^{-p} \geqslant n^2 \qquad (\cdot 414)$$

EXAMPLES VI (*a*)

1. If a, b, c are positive and $a+b \geqslant c$, prove that
$$\frac{a}{1+a} + \frac{b}{1+b} > \frac{c}{1+c}.$$

2. If x, y, z are real, show that
$$x^4+y^4+z^4 > y^2z^2+z^2x^2+x^2y^2.$$

3. If a, b, c be positive numbers, not all equal, prove that
(i) $$a^3+b^3+c^3 > 3abc.$$
(ii) $$\frac{a+b+c}{3} > (abc)^{\frac{1}{3}}, \text{ or } (a+b+c)^3 > 27abc.$$

4. Prove that if a, b, c be positive,
$$(b+c)(c+a)(a+b) \geqslant 8abc.$$
[**Hint.** $a+b \geqslant 2\sqrt{ab}$, etc.]

5. Prove that $\frac{a+x}{b+x} \gtrless \frac{a}{b}$, according as $b \gtrless a$, if x be positive; or according as $a \gtrless b$, if x be negative.

6. Prove that $\frac{a+x}{b+x}$ is nearer to 1 than $\frac{a}{b}$, x being positive.

[Hint. Show that $\left(\frac{a}{b}-1\right)^2 > \left(\frac{a+x}{b+x}-1\right)^2$.]

7. If $a^2+b^2=1$ and $x^2+y^2=1$, show that $ax+by<1$.

Hence show that all points on the circle $x^2+y^2=1$, lie on the same side of the line $ax+by-1=0$.

8. Show that $a^4+b^4>a^3b+b^3a$.

Sol. There is no loss of generality if we suppose $a>b$.

$\therefore \quad a^3>b^3$, or $a^3(a-b)>b^3(a-b)$.

9. If x, y, z be positive numbers, show that

$$(x+y+z)(yz+zx+xy)>9xyz,$$

unless x, y, z are all equal.

[Hint. Use A>G>H.]

10. If $m>n$, prove that $a^m+a^{-m} > a^n+a^{-n}$.

[Hint. a^m-a^n and $1-a^{-m-n}$ have the same sign, and the product is positive for all real values of a.]

11. Prove that, if a's and b's be real,

$$(a_1^2+a_2^2)(b_1^2+b_2^2) \geqslant (a_1b_1+a_2b_2)^2,$$

the sign of equality holding when $\frac{b_1}{a_1}=\frac{b_2}{a_2}$.

12. $b^2c^2+c^2a^2+a^2b^2 \geqslant abc(a+b+c)$.

13. If $a>b$, prove that $a^ab^b > a^bb^a$.

[Hint. $(a/b)^a > (a/b)^b$.]

14. Show that $a^4+b^4+c^4+d^4 \geqslant 4abcd$, where a, b, c, d are positive numbers.

15. Prove that, if a, b, c be positive,

$$bc(b+c)+ca(a+c)+ab(a+b) \geqslant 6abc.$$

[Hint. The inequality may be written as

$$a(b-c)^2+b(c-a)^2+c(a-b)^2 \geqslant 0.]$$

16. If x, y, z be positive numbers, not all equal, show that

$$(x+y+z)^3 > 27(y+z-x)(z+x-y)(x+y-z).$$

17. If A be the area and $2s$ the sum of the three sides of a triangle, prove that $\mathrm{A} \leqslant \frac{s^2}{3\sqrt{3}}$.

Sol. $(s-a)+(s-b)+(s-c)=s$.

$$\therefore \quad \left(\frac{s}{3}\right) > [(s-a)(s-b)(s-c)]^{\frac{1}{3}} \qquad (\mathrm{A}>\mathrm{G})$$

$$\therefore \quad \frac{s^3}{27} > (s-a)(s-b)(s-c),$$

or
$$\frac{s^2}{3\sqrt{3}} > \sqrt{s(s-a)(s-b)(s-c)} = \mathrm{A}.$$

18. If x, y, z are positive numbers such that $x+y+z=1$, then x^2yz cannot be greater than $\frac{1}{64}$. [Math. Trip. 1935]

19. Prove that $(1^r+2^r+\ldots+n^r)^n > n^n(n!)^r$

20. Show that $\frac{a_1}{a_2}+\frac{a_2}{a_3}+\cdots+\frac{a_{n-1}}{a_n}+\frac{a_n}{a_1} > n.$

[Hint. $\frac{1}{n}\left(\frac{a_1}{a_2}+\cdots+\frac{a_n}{a_1}\right) > \left(\frac{a_1}{a_2}\cdot\frac{a_2}{a_3}\ldots\frac{a_n}{a_1}\right)^{\frac{1}{n}} = 1.$]

6·5. Cauchy's Inequality. *If* $a_1, a_2, \ldots ; b_1, b_2 \ldots,$ *are two sets of real numbers, then*

$$\Sigma a_1^2 \, \Sigma b_1^2 \geqslant (\Sigma a_1 b_1)^2. \qquad (\cdot 5)$$

First proof.

$$(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2) = \Sigma a_1^2 b_1^2 + \Sigma a_1^2(b_2^2+b_3^2)$$
$$= (\Sigma a_1 b_1)^2 + \Sigma a_1^2(b_2^2+b_3^2) - 2\Sigma a_1 b_1 a_2 b_2 = (\Sigma a_1 b_1)^2 + \Sigma(a_1 b_2 - a_2 b_1)^2.$$

Hence $\Sigma a_1^2, \Sigma b_1^2 > (\Sigma a_1 b_1)^2$, unless $a_1 b_2 + a_2 b_1 = 0 = a_2 b_3 - a_3 b_2$ etc.

In the latter case $\frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3}$ and the sign of inequality in (·5) holds. The Theorem can obviously be generalised from the identity

$$(\Sigma a_i \, b_i)^2 = \Sigma a_i^2 \, \Sigma b_i^2 - \Sigma(a_1 b_2 - a_2 b_1)^2.$$

Second Proof. For every real value of λ,

$$\sum_{i=1}^{i=n} (\lambda a_i + b_i)^2 \equiv \mathrm{A}\lambda^2 + 2\lambda \mathrm{B} + \mathrm{C} \geqslant 0,$$

where $\mathrm{A} \equiv \Sigma a_1^2$, $\mathrm{B} \equiv \Sigma a_1^2 b_1^2$, and $\mathrm{C} \equiv \Sigma b_1^2$.

The sign of equality will hold only if, for every $i(=1, 2, \ldots n)$, $\lambda a_i + b_i = 0$, *i.e.*, $\frac{b_1}{a_1} = \frac{b_2}{a_2} = \cdots = \frac{b_n}{a_n} = -\lambda.$

In this case, obviously the sign of equality in (·5) also holds. As seen above, for every other real value of λ.

$$\mathrm{A}\lambda^2 + 2\lambda \mathrm{B} + \mathrm{B} > 0,$$

or
$$\frac{1}{\mathrm{A}}\left[(\mathrm{A}\lambda + \mathrm{B})^2 + \mathrm{AC} - \mathrm{B}^2\right] > 0.$$

A is +ve, and this is to be true for *all* real values of λ,

$$\therefore \quad \mathrm{AC} > \mathrm{B}^2.$$

Ex. Prove, without using the formula, that
$$(a^2+b^2+c^2)(x^2+y^2+z^2) > (ax+by+cz)^2.$$

6·51. *If $a_1, a_2 \ldots, a_n$, and $b_1\ b_2, \ldots, b_n$ are two sets of positive numbers, both arranged either in descending, or ascending order of magnitude, then*
$$\tfrac{1}{n}\Sigma a_1\ b_1 \geqslant (\tfrac{1}{n}\Sigma a_1)(\tfrac{1}{n}\Sigma b_1) \qquad (\cdot 511)$$

Illustration. Let us take three numbers in each set. As a_1-a_2, b_1-b_2 have the same sign,
$$(a_1-a_2)(b_1-b_2) \geqslant 0$$
or
$$a_1b_1+a_2b_2 \geqslant a_1b_2+a_2b_1. \qquad (\cdot 512)$$
Similarly
$$a_2b_2+a_3b_3 \geqslant a_2b_3+a_3b_2,$$
$$a_3b_3+a_1b_1 \geqslant a_3b_1+a_1b_3.$$
Adding,
$$2\Sigma a_1b_1 = \Sigma a_1b_2 = \Sigma a_1 \Sigma b_1 - \Sigma a_1b_1,$$
$$\therefore\ 3\Sigma a_1b_1 \geqslant \Sigma a_1 . \Sigma b_1.$$

Proof. If we write down ${}^nC_2 = \tfrac{1}{2}n(n-1)$ inequalities of the type (·512) which can be formed with the numbers $a_1, a_2 \ldots, a_n$ and the corresponding numbers $b_1, b_2 \ldots, b_n$, and add them up, we get
$$(n-1)\Sigma a_1b_1 \geqslant \Sigma a_pb_q. \qquad (p \neq q)$$
Now
$$\Sigma a_pb_q = \Sigma a_1\ \Sigma b_1 - \Sigma a_1b_1. \ .$$
Hence
$$(n-1)\ \Sigma a_1b_1 \geqslant \Sigma a_1\ \Sigma b_1 - \Sigma a_1b_1,$$
or
$$n\Sigma a_1b_1 \geqslant \Sigma a_1,\ \Sigma b_1,$$
i.e.,
$$\tfrac{1}{n}\ \Sigma a_1b_1 \geqslant (\tfrac{1}{n}\Sigma a_1).(\tfrac{1}{n}\Sigma b_1).$$

The sign of the inequality will change if one set is in ascending order and the other in descending order. As a particular case, if $a_1, a_2 \ldots a_n$ be in ascending order,

the set $\dfrac{1}{a_1}, \dfrac{1}{a_2}, \cdots \dfrac{1}{a_n}$ is in descending order, and $\Sigma a_1\ \Sigma \dfrac{1}{a_1} \geqslant n^2$. (·513)

6·6. *If x is any positive number different from unity, and p and q are positive rationals, then*
$$\frac{x^p-1}{p} > \frac{x^q-1}{q}, \text{ when } p > q. \qquad (\cdot 601)$$

Let m be a positive integer, then
$$\frac{x^{m+1}-1}{m+1} - \frac{x^m-1}{m} = \frac{x-1}{m+1}[x^m+x^{m-1}+\ldots+1]$$
$$-\frac{x-1}{m}[x^{m-1}+x^{m-2}+\ldots+1]$$
$$= \frac{x-1}{m(m+1)}[mx^m-(x^{m-1}+x^{m-2}+\ldots+1)]$$

$$= \frac{x-1}{m(m+1)}\left[(x^m-x^{m-1})+(x^m-x^{m-2})+\ldots+(x^m-1)\right]$$

$$= \frac{(x-1)^2}{m(m+1)}\left[x^{m-1}+x^{m-2}(x+1)+\ldots\right.$$
$$\left.+(x^{m-1}+x^{m-2}+\ldots+1)\right]$$
$$>0,$$

for each term within the brackets is positive and $(x-1)^2/m(m+1)$ is also positive. *Thus if n and m are positive integers, and $n>m$ and $y\neq 1$, then*

$$\frac{y^n-1}{n} > \frac{y^m-1}{m}.$$

Now put $y=x^{\frac{1}{a}}$, $p=n/a$, $q=m/a$, a being a positive integer; the above inequality then becomes

$$\frac{x^p-1}{p} > \frac{x^q-1}{q}, \qquad p>q,\ x\neq 1,$$

and the theorem is proved.

6·61. *If x and y are positive unequal numbers and p is any rational number except 1, then*

$$px^{p-1}(x-y) > x^p-y^p > py^{p-1}(x-y), \qquad (\cdot 611)$$

unless $0<p<1$, in which case

$$px^{p-1}(x-y) < x^p-y^p < py^{p-1}(x-y). \qquad (\cdot 612)$$

Proof. In (·601), put $q=1$, and replace x by x/y, we get

$$x^p-y^p \gtrless py^{p-1}(x-y), \text{ according as } p \gtrless 1. \qquad (\cdot 613)$$

In this relation interchange x and y,

then $$y^p-x^p \gtrless px^{p-1}(y-x) \quad p \gtrless 1,$$

or $$x^p-y^p \lessgtr px^{p-1}(x-y) \quad p \gtrless 1.$$

Thus $px^{p-1}(x-y)>x^p-y^p>py^{p-1}(x-y)$ if $p>1$,
and $px^{p-1}(x-y)<x^p-y^p<py^{p-1}(x-y)$, if $0<p<1$. (·614)

Now in (·614), put $p=q+1$, where $q>0$,
then $$(q+1)\ x^q(x-y)<x^{q+1}-y^{q+1}.$$

In this relation replace x and y by x^{-1}, y^{-1} respectively,
then $$(q+1)x^{-q}(x^{-1}-y^{-1})>x^{-q-1}-y^{-q-1},$$
or $$qx^{-q-1}-qx^{-q}y^{-1}>x^{-q}y^{-1}-y^{-q-1},$$
or $$qx^{-q-1}y-qx^{-q}>x^{-q}-y^{-q},$$
i.e., $$-qx^{-q-1}(x-y)>x^{-q}-y^{-q}.$$

This is the first part of (·611), when $-q$ (<0) is put equal to p. To prove the second part of (·611) for $p<0$, interchange x and y in the first part;

then $$py^{p-1}(y-x)>y^p-x^p.$$
$$\therefore x^p-y^p>py^{p-1}(x-y).$$

This completes the proof of the inequality.

6·62. In the inequalities (·611) and (·612), put $y=1$; we have then

$$px^{p-1}(x-1)>x^p-1>p(x-1), \text{ if } p>1 \text{ or } <0, \quad (\cdot 621)$$

and $$px^{p-1}(x-1)<x^p-1<p(x-1), \text{ if } 0<p<1, \ (x\neq 1). \quad (\cdot 622)$$

If in the second part of the inequalities x be replaced by $1+x$, the inequalities become

$$(1+x)^p>1+px, \quad \text{if } p>1 \text{ or } <0; \quad (\cdot 622)$$
$$(1+x)^p<1+px, \quad \text{if } 0<p<1, \ (x\neq 0). \quad (\cdot 623)$$

Changing x to $-x$, we have

$$(1-x)^p>1-px, \text{ if } p>1 \text{ or } <0; \quad (6\cdot 24)$$
$$(1-x)^p<1-px, \text{ if } 0<p<1. \quad (6\cdot 25)$$

6·63. *If $a_1, a_2, \ldots\ldots a_n$, be n positive numbers which are not all equal and p is any rational, except 0 and 1, then*

$$\frac{a_1^p+a_2^p+\ldots\ldots+a_n^p}{n} \gtrless \left[\frac{a_1+a_2+\ldots+a_n}{n}\right]^p \quad (\cdot 631)$$

according as p does not, or does, lie between 0 and 1.

Let $b_1, b_2, \ldots, b_n$ be a set of n numbers such that $b_1+b_2+\ldots+b_n=n$. Then

$$\frac{a_1}{b_1}=\frac{a_2}{b_2}=\ldots\ldots=\frac{a_n}{b_n}=\frac{\Sigma a_1}{n}.$$

Substituting for a's in the left-hand side of the inequality (·631), we see that we have to prove $b_1^p+b_2^p+\ldots\ldots+b_n^p \gtrless n$, according as p does not, or does, lie between 0 and 1.

Now from (·621) and (·622)

$$x^p-1 \gtrless p(x-1),$$

according as p does not, or does, lie between 0 and 1.

Replacing x by $b_1, b_2, \ldots\ldots, b_n$, respectively, and adding we get

$$\Sigma b_1^p-n \gtrless p(\Sigma b_1-n)=0. \quad [\text{From assumption}$$
$$\therefore \Sigma b_1^p>n, \quad \text{if } p>1, \text{ or } p<0.$$
and $$\Sigma b_1^p<n, \quad \text{if } 0<p<1,$$

This completes the proof.

6·64. *If* $a_1, a_2, \ldots\ldots a_n$; $p_1, p_2, \ldots\ldots, p_n$, *are two sets of* n *rationals, and those of the first set are not all equal, then*

$$\frac{p_1a_1^p+p_2a_2^p+\ldots\ldots+p_na_n^p}{p_1+p_2+\ldots\ldots+p_n} \gtrless \left(\frac{p_1a_1+p_2a_2+\ldots+p_na_n}{p_1+p_2+\ldots+p_n}\right)^p,$$

according as p, *which is rational, does not, or does lie between 0 and 1.*

Let $q_1, q_2, \ldots\ldots q_n$ be positive integers, such that $\frac{p_1}{q_1}=\frac{p_2}{q_2}=\cdots=\frac{p_n}{q_n}$, and $q_1+q_2+\ldots+q_n=m$, the inequality becomes

$$\frac{q_1a_1^p+q_2a_2^p+\ldots+q_na_n^p}{m} \gtrless \left(\frac{q_1a_1+\ldots+q_na_n}{m}\right)^p,$$

and this is a particular case of (·631) when q_1 a's are equal to a_1, q_2 to a_2, and so on.

6·7. Applications to maxima and minima.

(*a*) If $x, y, z, \ldots w$ are n positive variables such that $x+y+z+\ldots+w=k$, (a constant), $xyz\ldots w$ has the maximum value when $x=y=z\ldots\ldots=w$.

From (6·3) $\quad \frac{x+y+z+\ldots+w}{n} \geqslant (xyz\ldots w)^{\frac{1}{n}}$.

The sign of equality holds when $x=y=\ldots=w$. Thus the maximum value of

$$(xyz\ldots w)^{\frac{1}{n}}=\frac{k}{n}, \text{ or that of } xyz\ldots w=\left(\frac{k}{n}\right)^n.$$

(*b*) If $xyz\ldots w=c$, then $x+y+z\ldots+w$ is least if $x=y=\ldots=w$.

It also follows from (6·3) that $A=G$ if $x=y=\ldots=w$, and in general, $A>G$. Thus the least value of A is G when $x=y=\ldots\ldots=w$.

(*c*) If $x, y, z\ldots\ldots w$, are positive variables and $p, q, r, \ldots\ldots$are positive constants such that $\Sigma px=k$, then $x^py^q\ldots$has the maximum value $\left(\frac{k}{\Sigma p}\right)^{\Sigma p}$, when $x=y=\ldots=\frac{k}{\Sigma p}$.

(*d*) If $x, y, z, \ldots$ are positive variables and $p, q, \ldots$ are positive constants such that $x^pq^q\ldots=k$, then Σpx has a maximum value

$$\text{when } x=y=z=\ldots=k^{\frac{1}{\Sigma p}}.$$

These follow from (6·31).

(*e*) If $x^p+y^p+\ldots\ldots+w^p=k$, then according as p does not or does lie between 0 and 1, the greatest or the least value of $x+y+z+\ldots\ldots+w$ occurs, when $x=y=z\ldots\ldots=w$. This follows from (6·63).

EXAMPLES VI (b)

All the numbers involved in the following examples are positive rationals, unless otherwise stated.

If a, b, c, are all positive, prove that

1. $(a+b-c)^2+(b+c-a)^2+(c+a-b)^2>bc+ca+ab$.

2. $a\,(a-b)(a-c)+b\,(b-c)(b-a)+c\,(c-a)(c-b)>0$.

[If both sides of an inequality are symmetric functions of a_1, a_2,.....a_n, there is no loss of generality in assuming $a_1\geqslant a_2\geqslant a_3\ldots\ldots\geqslant a_n$. Here suppose $a>b>c$. Thus, $a\,(a-b)(a-c)>b\,(b-c)(a-b)$ and $c\,(c-a)(c-b)>0$].

3. $a^3+b^3+c^3+3abc>a^2(b+c)+b^2(c+a)+c^2(a+b)$.

4. $a^n(a-b)(a-c)+b^n(b-c)(b-a)+c^n(c-a)(c-b)>0$, when $n\geqslant 0$ or <-1.

5. If a and b are real, then
$$(a^2+b^2+1)^2\geqslant(a^2+b)^2+(b^2+a)^2+1.$$

6. Prove that
$$\left(1-\frac{a_1}{1+a_1}\right)\left(1-\frac{a_2}{1+a_2}\right)\ldots\ldots\left(1-\frac{a_n}{1+a_n}\right)$$
$$>1-\left(\frac{a_1}{1+a_1}+\frac{a_2}{1+a_2}+\frac{a_n}{1+a_n}\right),$$
where a_1, a_2,.....a_n are unequal positive quantities.

7. If x, y, z are positive numbers, such that $x+y+z=1$, then x^2yz cannot be greater than $\frac{1}{64}$. *(Math. Tripos, 1935)*

8. If x, y, z are positive quantities, such that $x+y+z=1$;

prove that (*i*) $\dfrac{x}{2-x}+\dfrac{y}{2-y}+\dfrac{z}{2-z}\geqslant\dfrac{3}{5}$,

(*ii*) $x^3y^2z\leqslant\dfrac{1}{432}$. *(St. Catherine, 1937)*

9. If a, b, c, d are positive numbers such that $a+b+c+d=3s$, prove that $abcd>81\,(s-a)(s-b)(s-c)(s-d)$.

Sol. $a=\dfrac{(s-b)+(s-c)+(s-d)}{3}>[(s-b)(s-c)(s-d)]^{\frac{1}{3}}$, with three more similar relations. On multiplying we get the result.

10. If a, b, c, are positive, prove that
$$\frac{2}{b+c}+\frac{2}{c+a}+\frac{2}{a+b}\geqslant\frac{9}{a+b+c}.$$

Sol. Put $a+b+c=s$, then the inequality to be proved becomes

$$\frac{s}{s-a}+\frac{s}{s-b}+\frac{s}{s-c} \geqslant \frac{9}{2}.$$

Now $\frac{1}{3}\left[\frac{s}{s-a}+\frac{s}{s-b}+\frac{s}{s-c}\right] \geqslant \frac{s}{[(s-a)(s-b)(s-c)]^{\frac{1}{3}}}$,

and $\frac{2}{3}=\frac{1}{3}\left[\frac{s-a}{s}+\frac{s-b}{s}+\frac{s-c}{s}\right] \geqslant \frac{[(s-a)(s-b)(s-c)]^{\frac{1}{3}}}{s}$.

Multiplying the two relations,

$$\frac{2}{9}\left[\frac{s}{s-a}+\frac{s}{s-b}+\frac{s}{s-c}\right] \geqslant 1.$$

11. If a, b, c, d are positive numbers, prove that

$$\frac{3}{b+c+d}+\frac{3}{c+d+a}+\frac{3}{d+a+b}+\frac{3}{a+b+c} \geqslant \frac{16}{a+b+c+d}.$$

12. If $s=a_1+a_2+\ldots\ldots+a_n$, show that

$$\frac{s}{s-a_1}+\frac{s}{s-a_2}+\ldots\ldots+\frac{s}{s-a_n}>\frac{n^2}{n-1},$$

$a_1, a_2, \ldots\ldots$being unequal positive numbers.

13. If a, b, c be unequal positive numbers, such that the sum of any two is greater than the third, show that

$$\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}>\frac{9}{a+b+c}.$$

[**Hint.** Put $a+b+c=2s$.]

14. If $a_1, a_2, \ldots\ldots a_n$ are unequal positive numbers such that $a_1+a_2+\ldots\ldots+a_n=(n-1)s$, prove that

(i) $a_1\, a_2 \ldots\ldots\, a_n>(n-1)^n\,(s-a_1)(s-a_2)\ldots\ldots(s-a_n)$,

(ii) $(n-1)^n s^n>n^n\, a_1 a_2 \ldots\ldots a_n$.

15. If a, b, c, d be in H. P. then $a+d>b+c$.

16. If a, b, c be in H. P. and $n>1$, then $a^n+c^n>2b^n$.

17. If a, b, c be in H. P, then

$$\frac{a+b}{2a-b}+\frac{c+b}{2c-b}>4. \qquad \textit{(Pembroke)}$$

Prove that

18. $(n+1)^n>2^n(n\,!)$, n being a positive integer.

19. $\left(\frac{mn+1}{m+1}\right)^{m+1}>n^m$, m being a positive integer.

[**Hint : Apply 6·3 to** $n, n, n\ldots\ldots n, 1$.]

If n is a positive integer, prove that

20. $2^n > 1+n\sqrt{2^{n-1}}$.

21. $(n\,!)^3 < n^n \left(\frac{n+1}{2}\right)^{2n}$. *(St. Catherine)*

22. $a^n - b^n > n(a-b)(ab)^{\frac{n-1}{2}}$.

23. If $a_1 > a_2 > a_3 \ldots > a_n$, then
$(a_1-a_n)^{n-1} > (n-1)^{n-1}\ (a_1-a_2)\ a_2-a_3)\ldots\ldots(a_{n-1}-a_n)$.

24. If $x+y=2a$, prove that $x^m+y^m > 2a^m$.

If $a_1,\ a_2,\ldots;\ b_1,\ b_2,\ldots$ are two sets of positive numbers, prove that

25. $$(a_1b_1+a_2b_2+a_3b_3+\ldots+a_nb_n)\left(\frac{a_1}{b_1}+\frac{a_2}{b_2}+\ldots+\frac{a_n}{b_n}\right) \geqslant (a_1+a_2+\ldots+a_n)^2.$$

26. $$\left(\frac{a_1}{b_1}+\frac{a_2}{b_2}+\ldots+\frac{a_n}{b_n}\right)\left(\frac{b_1}{a_1}+\frac{b_2}{a_2}+\ldots+\frac{b_n}{a_n}\right) \geqslant n^2.$$

If a, b, c are positive real numbers, prove that

27. $(a+b+c)(a^2+b^2+c^2) \geqslant 9abc$.

28. $a^4+b^4+c^4 > abc\,(a+b+c)$.

29. $$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geqslant \frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}+\frac{1}{\sqrt{ab}}.$$

30. Prove that $a^5+b^5+c^5+d^5 \geqslant abcd\,(a+b+c+d)$, a, b, c, d being positive numbers.

31. If $a_1,\ a_2 \ldots a_n$ are positive numbers, prove that
$a_1^{n+1}+a_2^{n+1}+\ldots+a_n^{n+1} > a_1a_2\ldots a_n(a_1+a_2+\ldots+a_n)$.

32. Show that $\left(\frac{x^2+y^2+z^2}{x+y+z}\right)^{x+y+z} > x^xy^yz^z$.

[Hint : Use 6·31].

33. If a and b are positive and $a+b=1$, show that

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2 \geqslant \frac{25}{2}.$$ *[Math. Tripos, 1926]*

[Solution. $\frac{a+b}{2} \geqslant \sqrt{ab}$, *i.e.*, $\frac{1}{4} \geqslant ab$, and $4 \leqslant \frac{1}{ab}$.

Now $\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2 = a^2+b^2+\frac{1}{a^2}+\frac{1}{b^2}+4$

$$= (a+b)^2-2ab+\frac{(a+b)^2-2ab}{a^2b^2}+4 = -2ab+\frac{1}{a^2b^2}-\frac{2}{ab}+5.$$

Thus we have to prove that

$$\frac{1}{a^2b^2}-\frac{2}{ab}+1 \geqslant 9, \text{ or that } \left(\frac{1}{ab}-1\right)^2 \geqslant 9.$$

Now $\because \quad \frac{1}{ab} \geqslant 4, \quad \left(\frac{1}{ab}-1\right)^2 \geqslant 9.$]

34. If $a+b=4$, and a and b are positive, show that

$$a^2+\frac{1}{a^2}+b^2+\frac{1}{b^2} \geqslant 8\tfrac{1}{2}. \qquad \textit{(Selwyn 1928)}$$

35. Prove that $\frac{1}{2\sqrt{n+1}} < \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$.

[Let $u_n = \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n}$, $\therefore u_n < \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n+1}$.

Multiplying, $u_n^2 < \frac{1}{2n+1}$, or $u_n < \frac{1}{\sqrt{2n+1}}$.

Also $(2n+1)u_n = \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} \cdot (2n+1)$

$$= \frac{3}{2} \cdot \frac{5}{4} \cdots \frac{2n+1}{2n}$$

$$> \frac{4}{3} \cdot \frac{6}{5} \cdots \frac{2n+2}{2n+2}.$$

Multiplication gives

$$(2n+1)^2 u_n^2 > (n+1), \text{ or } u_n^2 > \frac{n+1}{(2n+1)^2}.$$

Hence $u_n < \frac{1}{\sqrt{2n+1}}$ and $u_n > \frac{\sqrt{n+1}}{2n+1} > \frac{1}{2\sqrt{n+1}}$.

36. Prove that if n be a positive integer,

$$n^n \geqslant 1 \cdot 3 \cdot 5 \ldots\ldots (2n-1) \geqslant (2n-1)^{\frac{n}{2}},$$

and that $(1+\frac{1}{2})(1+\frac{1}{4}) \ldots\ldots \left(1+\frac{1}{2n}\right) < (2n+1)$. [*King's 1931*]

37. Show that $2(a^2+b^2+c^2)(a+b+c) > a^3+b^3+c^3+15abc$, if a, b, c are unequal positive quantities.

[Hint. $2\Sigma a^2 \Sigma a - \Sigma a^3 - 15\,abc = \Sigma a(\Sigma a^2 - \Sigma bc) + 2\Sigma a(b-c)^2 > 0$.]

38. Show that if n be a positive integer,

$$(n+1)^{2n+1} < n^n (n+2)^{n+1}.$$

[In the second part of (·621) put $x = \frac{n+2}{n+1}$, $p = \frac{n+1}{n}$.]

39. If a, b, c are positive and $a+b+c=1$, then

$$\frac{8}{27\,abc} > \left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right) \geqslant 8.$$

[Hint. $\Sigma(1-a)=2$, $\therefore$ by (6·3), $(1-a)(1-b)(1-c) < \frac{8}{27}$.

Again $\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right)\geqslant 8$, implies $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geqslant 9$, which follows from (·413), when $p=1$, or from (·513).]

40. Prove that $\frac{a^{\frac{1}{5}}+b^{\frac{1}{5}}}{2} > \left(\frac{a+b}{2}\right)^{\frac{1}{5}}$.

41. If a and b are unequal, and p and q are two numbers, such that $\frac{1}{p}+\frac{1}{q}=1$, then

$$a^{\frac{1}{p}}\, b^{\frac{1}{q}} < \frac{a}{p}+\frac{b}{q}.$$

[If $m<1$, $x^m-1<m(x-1)$, (·622). Put $x=\frac{a}{b}$, $m=\frac{1}{p}$,

then $\frac{a^{\frac{1}{p}}}{b^{\frac{1}{p}}} < \frac{1}{p}\frac{a-b}{b}+1=\frac{1}{p}\frac{a}{b}+1-\frac{1}{p}=\frac{1}{p}\frac{a}{b}+\frac{1}{q}$, $\because \left(\frac{1}{q}=1-\frac{1}{p}\right)$.

Hence $a^{\frac{1}{p}} b^{\frac{1}{q}} < \frac{a}{p}+\frac{b}{q}$.]

42. Prove that $xy<\frac{x^p}{p}+\frac{y^q}{q}$.

43. If a_1, a_2, b_1, b_2 are positive numbers, then

$$a_1 b_1+a_2 b_2 \leqslant (a_1^p+a_2^p)^{\frac{1}{p}}(b_1^q+b_2^q)^{\frac{1}{q}},$$

where p and q are positive numbers such that $\frac{1}{p}+\frac{1}{q}=1$.

[In Ex. 42, replace x and y by

$$\frac{a_1}{(a_1^p+a_2^p)^{\frac{1}{p}}}, \frac{b_1}{(b_1^q+b_2^q)^{\frac{1}{q}}} \text{ and } \frac{a_2}{(a_1^p+a_2^p)^{\frac{1}{p}}}, \frac{b_2}{(b_1^q+b_2^q)^{\frac{1}{q}}}$$

respectively.

$$\therefore \quad \frac{a_1 b_1}{AB} \leqslant \frac{1}{p}\frac{a_1^p}{A^p}+\frac{1}{q}\frac{b_1^q}{B^q}, \quad \frac{a_2 b_2}{AB} \leqslant \frac{1}{p}\frac{a_2^p}{A^p}+\frac{1}{q}\frac{b_2^q}{B^q},$$

where $A=(a_1^p+a_2^p)^{\frac{1}{p}}$, $B=(b_1^q+b_2^q)^{\frac{1}{q}}$. Addition gives

$$\frac{a_1 b_1+a_2 b_2}{AB} \leqslant 1.]$$

44. If p and q are positive such that $\frac{1}{p}+\frac{1}{q}=1$, and $a_1, a_2,\ldots a_n$ $b_1, b_2,\ldots b_n$ are two sets of positive numbers, then

$$a_1b_1+a_2b_2+\ldots+a_nb_n \leqslant (a_1^p+a_2^p+\ldots+a_n^p)^{\frac{1}{p}}\ (b_1^q+b_2^q+\ldots+b_n^q)^{\frac{1}{q}}.$$

[*Hint.* Proceed as in Ex. 43.; or as follows :—
Let $S=\Sigma a_1b_1$, $\Sigma a_1^p=U^p$, $\Sigma b_1^p=V^p$, then by Ex. 41

$a_1b_1=\lambda a_1.\lambda^{-1}b_1 \leqslant \lambda^p \frac{a_1^p}{p}+\lambda^{-q}\frac{b_1^q}{q}$. Adding n such inequalities,

$S \leqslant \frac{\lambda^p}{p}\ U^p+\frac{\lambda^{-q}}{q}V^q$. Choose λ so that $\lambda^p\ U^p=\lambda^{-q}V^q=t$;

$$\therefore\quad S \leqslant t\left(\frac{1}{p}+\frac{1}{q}\right), \text{ or } S \leqslant t.$$

But $t=t^{\frac{1}{p}}\ .\ t^{\frac{1}{q}}=\lambda U.\lambda^{-1}V=UV$. Hence $S \leqslant UV$.]

45. If $x_1, x_2,\ldots x_n$ are positive numbers such that $\Sigma x=n$, prove that $\Sigma x^m \geqslant n$ if $m>1$. [Use ·621]

46. If $a_1, a_2,\ldots a_n$ are all positive and $s_n=a_1+a_2+\ldots+a_n$, then $(1+a_1)(1+a_2)\ldots(1+a_n) \leqslant 1+s_n+\frac{s_n^2}{2!}+\ldots+\frac{s_n^n}{n!}$.

[Let $U_n=1+s+\frac{s_n^2}{2!}+\ldots+\frac{s_n^n}{n!}-(1+a_1)(1+a_2)\ldots(1+a_n)$.

$$U_n-U_{n-1}=(s_n-s_{n-1})+\frac{s_n^2-s_{n-1}^2}{2!}+\ldots+\frac{s_n^{n-1}-s_{n-1}^{n-1}}{(n-1)!}+\frac{s_n^n}{n!}$$
$$-a_n(1+a_1)\ldots(1+a_{n-1}),$$

But $s_n^r-s_{n-1}^r>r\ a_n s_{n-1}^{r-1}$ by (·611).

$$\frac{U_n-U_{n-1}}{2a_n}>1+s_{n-1}+\frac{s_{n-1}^2}{2!}+\cdots+\frac{s_{n-1}}{(n-1)!}-(1+a_1)\ldots(1+a_{n-1})$$

$\therefore\ U_n-U_{n-1}>a_n\ U_{n-1}$, or $U_n>(1+a_n)U_{n-1}$; thus if $U_{n-1}>0$, so is U_n. But $U_2=1+s_2+\frac{s_2^2}{2}-(1+a_1)(1+a_2)=\frac{a_1^2+a_2^2}{2}>0$.

Hence $U_n>0$.]

47. Prove that if

$$\phi_n=\left(1+\frac{1}{n}\right)^n,\ \psi(n)=\left(1-\frac{1}{n}\right)^{-n},$$

then $\phi(n+1)>\phi(n)$ and $\psi\ (n+1)<\psi(n)$. Prove also that $\psi\ (n)>\phi(n)$ for all values of $n>1$, and deduce that $\phi(n)$ and $\psi\ (n)$ tend to a common limit as $n\to\infty$.

Proof. In (·611), put $x=1+\frac{1}{n+1}$, $y=1$, $p=\frac{n+1}{n}$;

then $$\left(1+\frac{1}{n+1}\right)^{\frac{n+1}{n}} - 1 > \frac{n+1}{n} \cdot \frac{1}{n+1} = \frac{1}{n},$$

or $$\left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^{n}, \text{ i.e., } \phi(n+1) > \phi(n).$$

Again, in the first part of (·611), put $x=1$,

$$y=1-\frac{1}{n+1}, \quad p=\frac{n+1}{n},$$

then $$\frac{n+1}{n} \cdot \frac{1}{n+1} > 1-\left(1-\frac{1}{n+1}\right)^{\frac{n+1}{n}},$$

or $$\left(1-\frac{1}{n}\right)^{-n} > \left(1-\frac{1}{n+1}\right)^{-n+1}, \text{ i.e., } \psi(n) > \psi(n+1).$$

Hence putting $n=2$, we get $4 > \psi(n)$, for $n>2$.

Also $$1-\frac{1}{n^2} < 1, \therefore \left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right) < 1,$$

or $$\left(1+\frac{1}{n}\right)^{n} < \left(1-\frac{1}{n}\right)^{-n}, \text{ i.e., } \psi(n) > \phi(n).$$

Again, $$\psi(n)-\phi(n)=\left(\frac{n}{n-1}\right)^{n}-\left(\frac{n+1}{n}\right)^{n}.$$

In (·611) put $x=\frac{n}{n-1}$, $y=\frac{n+1}{n}$, $p=n$,

then $$\left(\frac{n}{n-1}\right)^{n}-\left(\frac{n+1}{n}\right)^{n} < n\left(\frac{n}{n-1}\right)^{n-1}\left(\frac{n}{n-1}-\frac{n+1}{n}\right)$$

$$=\frac{1}{n}\psi(n) < \frac{4}{n}, \quad \because \psi(n) < 4.$$

Thus $\phi(n)$ and $\psi(n)$ tend to a common limit which is denoted by e.

CHAPTER VII

CONVERGENCE OF INFINITE SERIES

7·1. We have already pointed out in the first chapter that all the rational and irrational numbers, positive or negative (zero included), constitute **real numbers.**

Any seclection of real numbers, each separately specified, or the whole selection defined by a formula or formulae, is called a set. If the members of the set are endlessly numerous, we have an **infinite set.**

The positive integers constiute an infinite set. The rational numbers between 1 and 2, the real numbers between any two real numbers etc. are all infinite sets.

A Sequence (or Suite). *An infinite sequence*

$$x_1, x_2, x_3 \ldots\ldots, x_n \ldots$$

is an infinity of numbers specified in a definite order, by some assigned law, whereby a definite number x_n of the set can be associated with the corresponding positive integer n.

In general, x_n can be represented as a function $\phi(n)$ of a positive integer n, and by giving to n a suitable value, an arbitrarily selected number of the set is obtainable as the corresponding value of $\phi(n)$.

Three different notations are used for a sequence

$$x_1, x_2, \ldots, x_n \ldots$$
$$x_n \ (n=1, 2, \ldots)$$
$$(x_n).$$

Order is a part of the definition of a sequence. A formula $\phi(n)$ in which n takes successively all positive integral values in their natural order, completely determines a sequence.

A sequence (x_n) is said to be bounded if there exists a number k such that $x_n < k$ to every n.

Examples of Sequences

Let $\phi(n)=\frac{1}{n}$, then we have the sequence

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4} \ldots \frac{1}{n}, \ldots$$

whose terms are in H. P.

If $\phi(n)=r^n$, we have the sequence

$$r, r^2 r^3, \ldots r^n \ldots,$$

whose terms are in G. P.

The law of formation of x_n may be quite arbitrary; it need not, in particular, be embodied in any explicit formula enabling us to obtain x_n, for a given n, by direct calculation. For example,

(1) x_n=the decimal fraction for $\sqrt{2}$, terminated at the nth digit.

(2) x_n=the nth prime number, *i.e.*, the sequence 2, 3, 5, 7, 11...

(3) $(x_n)=1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \ldots$

$$\text{in which } x_n=\left(1+\tfrac{1}{2}+\cdots+\frac{1}{n}\right).$$

In (1) and (2), clearly no formula for $\phi(n)$ can be immediately written down.

The function $\phi(n)$ may be specified by an algebraical, trigonometrical, or any other analytical formula; or may be defined by verbal rules. The essential thing is that for every value of n, there shall be a rule giving a unique value for the nth term of the sequence.

7·12. The sequence (x_n) is said to increase steadily or to decrease steadily, according as $x_{n+1} \geqslant x_n$, or $x_{n+1} \leqslant x_n$, for *all* values of n. Both increasing and decreasing sequences are included in the the term **monotonic sequence.**

Definition 1. The sequence (x_n) is said to tend to a limit l, if given a positive number ϵ, however small, an index m can be found, such that $|x_n-l| < \epsilon$ for all values of $n \geqslant m$.

The above statement can be concisely written by the introduction of the following notation, a sort of mathematical shorthand. The above definition in the new notation is $\epsilon > 0$; $\exists m. \ |x_n-l| < \epsilon$, when $n \geqslant m$.

Symbolically $\epsilon > 0$ stands for "given a positive number ϵ, however small."

$\exists m$, stands for "an index m can be found," or "there exists an index m."

"." stands for 'such that'.

The fact of the approach to a limit may be stated as follows:— $|l-x_n| < \epsilon$ for *all* x's subsequent to some x, where ϵ is a *positive* number which may be assigned at pleasure, and must be

assigned before the attempt be made to find m, *i.e.*, an x such that for all subsequent x's the relation $| x_n - l | < \epsilon$ holds.

If a sequence has a *finite definite* limit, it is called a **convergent sequence.**

Def. II. The sequence (x_n) is said to tend to $+\infty$ with n if, given a number G, however large, we can choose an index m, so that

$$x_n > \mathrm{G} \qquad \text{for } \textit{every } n \geqslant m,$$

or, symbolically

$$\mathrm{G} > 0 \; ; \; \exists m. \quad x_n > \mathrm{G}, \text{ when } n \geqslant m.$$

We then write $x_n \to \infty$.

Similarly we define $x_n \to -\infty$.

If a sequence tends to *infinity* it is called a **divergent sequence.**

Def. III. If the conditions of none of the two preceding definitions are satisfied, *i.e.*, if x_n does not tend to a finite definite limit, nor to $+\infty$ nor to $-\infty$, (x_n) is said to oscillate as $n \to \infty$.

If, for all values of n, $| x_n | < k$, where k is some constant, (x_n) is said to oscillate finitely; otherwise infinitely. Such a sequence is called an **oscillary sequence.**

Oscillation is, according to its definition, a purely negative quality—a sequence oscillates, when it does not do certain other things.

7·13 We may assume that a *monotonic sequence has always a limit, either finite or infinite.*

The graphical representation makes it intuitively obvious. But we may add the following proof.

(1) Let us consider an increasing monotonic sequence, (x_n), such that if k be a finite number, independent of n, so that for every n,

$$x_n < k,$$

then x_n approaches a fixed limit λ such that

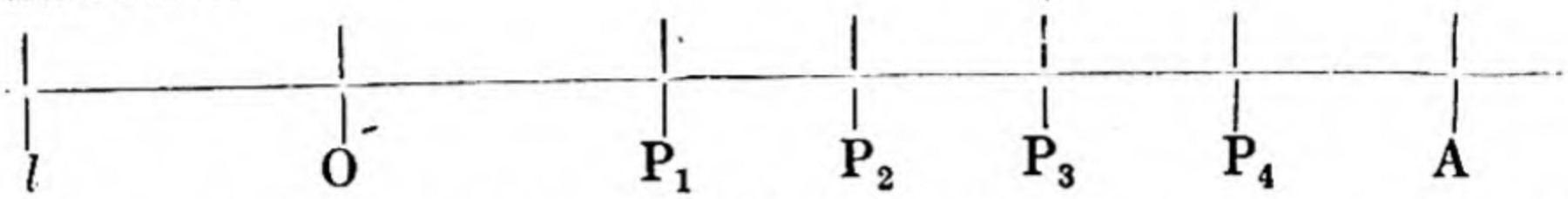
$$\lambda \leqslant k.$$

Let us represent the sequence of numbers x_n by the sequence of points P_n on a line l, such that the distance OP_n of P_n from a fixed point O on the line l equals the number x_n, for every n. The sequence of points

$$\mathrm{P}_1, \mathrm{P}_2, \ldots$$

is such that each point is to the right of the preceding one, but

all lie to the left of some fixed point A which represents the number k.



Now every point of the line l belongs to one or other of two mutually exclusive classes : either it has points of the sequence (P_n) to the right of it, or it has not. Moreover, every point of the first class is to the left of every point of the second class. Hence there must be some point P which separates the two classes, *i.e.*, every point to the left of P belongs to the first class, and every point to the right of P belongs to the second class. Since A belongs to second class, it is to the right of P. Hence if we put $OP=\lambda$, then $x_n \to \lambda < A$.

The sequence (x_n) has an upper limit λ, *i.e.*, a limit which exceeds any one of the numbers

$$x_1, x_2, x_3 \ldots x_n \ldots$$

(2) If (x_n) be an increasing monotonic sequence, and G a number, however large, so that $x_n > G$ for every $n > n_0$, then

$$x_n \to \infty,.$$

In the present case, taking the above illustration, the points of the sequence will steadily move towards the right, and how so far we may move along the line l, we will find points of the sequence to the right of the points of l, which is the statement made above.

(2) If (x_n) be a decreasing monotonic sequence, and k a finite number, such that for every n, $x_n \geqslant k$, then x_n approaches a finite limit $\lambda \geqslant k$.

(4) If (x_n) be a decreasing monotonic sequence, and G a number, however large, so that $x_n < -G$, for every $n > n_0$, then $x_n \to -\infty$. (3) and (4) can be illustrated in a manner similar to that for (1) and (2).

Examples VII (*a*)

1. Examine the convergence of each of the following sequences :—

(*i*) $x_n=2n$.

(*ii*) $x_n=\dfrac{1}{n+(-1)^n}$.

(*iii*) $x_n=(-1)^n$.

(*iv*) $x_n=1+\dfrac{1}{n}$.

(*v*) $x_n=r$ (r^n being independent of n).

(*vi*) $x_n=n[1+(-1)^n]$.

(2.) Determine the least value of m for which it is true that (*a*) $n^2+(-1)^n\ n>$ G. (*b*) $n^2+n>100$.

7·14. If u_n is a function of n, defined for all values of n, and if we add up the values of u_r for $r=1, 2, \ldots n$, we obtain another function of n, *viz.*

$$U_n=u_1+u_2+u_3\ldots\ldots+u_n=\sum_{r=1}^{n} u_r\ ,$$

also defined for all values of n.

When n is allowed to tend to infinity, we obtain an infinite series, whose terms are formed according to a given law.

The word 'infinite' before 'sequence' or 'series' simply means that *every* term in the sequence or the series is followed by another term.

Definition. The infinite series

$$u_1+u_2+\ldots\ldots+u_n+\ldots$$

is said to converge, diverge, or oscillate (finitely or infinitely), according as the sequence

$$U_1,\ U_2,\ldots U_n,\ldots$$

where $U_n=u_1+u_2+\ldots\ldots+u_n$

converges, diverges, or oscillates (finitely or infinitely).

If $U_n\to U$ then U is said to be the sum of the series.

Ex. 1. Consider the series

$$1+\frac{1}{5}+\frac{1}{5^2}+\cdots+\frac{1}{5^n}+\cdots$$

Here
$$s_n=\frac{1-\frac{1}{5^n}}{1-\frac{1}{5}}$$

$$\therefore\ s=\operatorname*{Lt}_{n\to\infty} s_n=5/4.$$

Ex. 2. Consider the series $1^2+2^2+3^2+\ldots+n^2+\ldots$

Here
$$s_n=\frac{n(n+1)(2n+1)}{6}.$$

$$\therefore\ s_n\to\infty\ ,\ \text{when}\ n\to\infty\ .$$

Ex. 3. Consider the series

$$11-7-4+11-7-4+11-7-4+\ldots$$

Here
$$\operatorname*{Lt}_{n\to\infty} s_n=0,\ \text{or}\ 11,\ \text{or}\ 4.$$

Examples VII (*b*)

1. Sum to infinity the series

$$\frac{2}{5}+\frac{7}{5^2}+\frac{2}{5^3}+\frac{7}{5^4}+\frac{2}{5^5}+\dots$$

2. Examine the convergence of the series :—

$$\frac{5}{7}+\frac{6}{7^2}+\frac{5}{7^3}+\frac{6}{7^4}+\dots$$

3. Show that the following series oscillates :—

$$13-7-4-2+13-7-4-2+\dots$$

4. Examine the convergence of the series

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\dots+\frac{1}{(n+1)(n+2)}+\dots$$

5. Show that the series

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots+\frac{(-)^{n-1}}{2n-1}+\dots$$

converges.

6. Examine the convergence of the series

$$1-3+5-7+9-\dots$$

7. Show that $\dot{7}=\frac{7}{9}$.

8. Prove that the sum to infinity of the series

$$\frac{1}{4}+\frac{1.2}{4.5}+\frac{1.2.3}{4.5.6}+\frac{1.2\,3.4}{4.5.6.7}+\dots \text{ is } \frac{1}{2}.$$

9. If n is a positive integer and $1+x>0$ (excluding $x=0$), prove that $(1+x)^n>1+nx$.

[$(1+x)^2=1+2x+x^2$ and $x^2>0$, $\therefore$ $(1+x)^2>1+2x$;

$\therefore$ $(1+x)^3>(1+2x)(1+x)=1+3x+2x^2>1+3x$, etc.]

10. If $r>1$ and G any given positive number, however large, show how to find a positive integer m such that $r^n>\mathrm{G}$ for all $n>m$.

[Let $r=1+x$; x is evidently positive and $(1+x)^n>1+nx$.

$\therefore$ $r^n>1+nx$. Now $1+nx>\mathrm{G}$, if $n>\frac{\mathrm{G}-1}{x}=\frac{\mathrm{G}-1}{r-1}$;

hence m is the greatest integer in $\frac{\mathrm{G}-1}{r-1}$ etc.]

(11). If $0<r<1$ and ϵ any given positive number, however small, show how to find a corresponding positive integer m such that $r^n < \epsilon$ for all $n>m$.

$$\left[\text{Since } \frac{1}{r}>1 \text{ and } \frac{1}{\epsilon}=G, \text{ apply Ex. 10.}\right]$$

7·15. There are certain series whose convergence can be inferred easily. Some of them have been treated in the above set of examples.

We consider now the geometric series which we shall need in subsequent work.

7·16. The infinite geometric series

$$1+r+r^2+\ldots\ldots+r^n+\ldots$$

(*a*) *converges if* $-1<r<1$, *and its sum is* $(1-r)^{-1}$,
(*b*) *diverges if* $r\geqslant 1$,
(*ci*) *oscillates finitely if* $r=-1$,
(*cii*) *oscillates infinitely if* $r<-1$.

Let r be any number different from unity, then

$$S_n=\frac{1-r^n}{1+r}=\frac{1}{1-r}-\frac{r^n}{1-r}, \quad \text{or} \quad S_n=\frac{r^n-1}{r-1}=\frac{r^n}{r-1}-\frac{1}{r-1}.$$

The convergence or divergence then depends upon r^n.

(*a*) Suppose $0<r<1$. [$\because$ $0<r<1$.

Now $r^{n+1}=r^n.\ r<r^n$

The sequence (r^n) is, therefore, a decreasing sequence and as r^n is positive, r^n tends to a limit, say l, then the relation $r^{n+1}=r^n.r$ gives

$$l=lr, \text{ when } n\to\infty$$
$$\therefore \quad l=0.$$

Thus the sequence (r^n) tends to zero as $n\to\infty$, when $0<r<1$, (see also Ex. 11, VII (*a*)).

If $-1<r<0$, put $\rho=-r$, so that $0<\rho<1$, and $r^n=(-1)^n\,\rho^n$. But $\rho^n\to 0$ as $n\to\infty$, $\therefore$ $r^n\to 0$ as $n\to 0$,

Hence the sequence S_n converges to $(1-r)^{-1}$. The series also, therefore, converges and has the sum $(1-r)^{-1}$.

(*b*) Suppose $r>1$. Now $r^{n+1}=r.\ r^n>r^n$ ($\because$ $r>1$).

The sequence (r^n) is an increasing sequence, and therefore either tends to infinity or to some limit (l say) which is obviously not zero, and is greater than 1.

If $r^n\to l$, then relation $r^{n+1}=r^n.\ r$ gives

$$l=lr, \qquad \therefore \quad l=0,$$

which is obviously not true. Hence $r^n\to\infty$.

[See also Ex. 10, VII (*a*)].

If $r=1$, $S_n=n$, hence $S_n\to\infty$ as $n\to\infty$.

Thus when $r \geqslant 1$, the sequence (S_n) is divergent, and therefore the series also diverges.

(*c*) If $r=-1$, the series becomes

$$1-1+1-1+1-1+\ldots$$

$S_{2n}=0$, $S_{2n+1}=1$, thus the series oscillates finitely between 0 and 1.

(*d*) Let $r<-1$. Put $r=-\rho$, so that $1<\rho$, and $r^n=(-1)^n\rho^n$.

Now $\rho^n \to \infty$ with n, hence $S_{2n} \to \infty$, $S_{2n+1} \to -\infty$.

Hence the series oscillates infinitely, if $-1<r$.

We now proceed to establish some of the criteria of convergence and divergence for infinite series.

General Test of Convergence

7·2. *The necessary and sufficient condition for the convergence of a series of real terms,* $\sum_{n=1}^{\infty} u_n$ *is that it must be possible to find* m *(a value of* n*), corresponding to any given positive number* ϵ*, however small, such that for all values of* $n \geqslant m$,

$$|s_n - s_m| < \epsilon,$$

i.e.,

$$|u_{m+1}+u_{m+2}+\ldots+u_{m+p}| < \epsilon,$$

no matter how large p *may be.*

The proof of the theorem rests on a definition of irrational numbers which is beyond the scope of this book.

General Theorem on Infinite Series

7·21. (*a*) *The convergence, or divergence, or oscillation of a series is not affected by the addition, omission, or alteration of a finite number of its terms.*

For, the sum of a finite number of terms is finite and definite, and that sum alters $\operatorname{Lt}_{n\to\infty} u_n$ merely by a finite definite number; so that if the series were originally convergent, divergent, or oscillatory, it would remain so.

Note that the *behaviour of the series near its beginning is of no consequence* in regard to its convergence, divergence, or oscillation. It is the property of $u_{m+1}+u_{m+2}+\ldots$ which is important. The question of convergence depends entirely upon the infinity of terms in the endlessly remote part of the

series corresponding to endless increase of m. A test of convergence need. therefore, have to do only with terms *after some definite term, i.e.* corresponding to *all values of n after some fixed value of n.*

7·22. Telescopic Series. Such series are of the form

$$s \equiv (u_1-u_2)+(u_2-u_3)+(u_3-u_4)+\ldots$$

In this case, $s_n=(u_1-u_2)+\ldots+(u_n-u_{n+1})$

$$=u_1-u_{n+1}.$$

Thus the terms of any s_n cancelling out in pairs, s_n reduces to only two terms and so shuts up like a telescope.

Thus *a telescopic series is convergent when Lt* u_n *exists.*

Again, let $V=v_1+v_2+v_3+\ldots$ be any series.

Then $v_n=V_n-V_{n-1}$, $V_0=0$.

Hence $V=(V_1-V_0)+(V_2-V_1)+(V_3-V_2)+\ldots$

This shows that *any series can be written as a telescopic series.*

This fact is of immense use in studying the general theory of series.

Ex. (1) Show that $\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\cdots$ converges to $\frac{1}{6}$.

Ex. (2) Show that $\sum_1^\infty \frac{a_n}{(1+a_1)\ldots(1+a_n)}$ converges.

Positive-Term Series

7·3. *A positive-term series either converges, or else diverges to* $+\infty$.

$\because$ U_n is an increasing monotonic sequence, it therefore tends to a finite limit or to infinity.

7·31. *If* u_n *is a positive function of n and* Σu_n *converges, then* $u_n \to 0$ *as* $n \to \infty$.

Let $U_n \equiv u_1+u_2+\ldots+u_n$

As the series converges, $U_n \to$ a limit U, *i.e.*,

$$\epsilon>0\ ;\ \exists\ m.\ |\ U_n-U\ |<\epsilon/2 \to \text{for } n>m.$$

$$\epsilon>0,\ \exists m'.\ |\ U_{n-1}-U\ |<\frac{\epsilon}{2} \text{ for } n-1>m'$$

Now $|\ u_n\ | = |\ U_n-U_{n-1}\ | = |\ U_n-U+U-U_{n-1}\ |$

$$\leqslant |\ U_n-U\ | + |\ U-U_{n-1}\ |$$

$<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$, whenever n is greater than the greater of m and m'.

Hence $u_n \to 0$ as $n \to \infty$.

If *follows from the above, that if* u_n *increases or tends to a limit, the series* Σu_n *cannot converge.*

7·32. *(i) A convergent series of positive terms remains convergent when each term is multiplied by a positive factor a_n, where $a_n<k$.*

(ii) If Σu_n is a divergent positive-term series, then so is $\Sigma a_n u_n$, if the factor $a_n>k$, where k is a positive constant.

Let $U_n=u_1+u_2+\ldots+u_n$ and $s_n=a_1u_1+a_2u_2+\ldots+a_nu_n$. As the U-series converges, the sequence (U_n) converges to a limit U.

Now $s_{n+1}-s_n=a_{n+1}\,u_{n+1}>0$, and $s_n<kU_n>k(U+\epsilon)$, ϵ being a positive small number.

The sequence (s_n) is an increasing sequence and $s_n<k(U+\epsilon)$ for all values of n. Hence s_n tends to s and $s\leqslant kU$.

To prove the second part we note that if $G(>0)$ be arbitrarily chosen, then the partial sums of Σu_n from a suitable index onwards are all $>G/k$.

From the same index onwards, the partial sums of $\Sigma a_n u_n$ are $>G$. Thus $\Sigma a_n u_n$ is divergent.

Ex. 1. Show that if $r<1$, the series $\sum_0^\infty r^n$ converges.

Ex. 2. If $z_1, z_2, z_3,\ldots z_n\ldots$ denote any digits, *i.e.* if each one of them be one of the numbers 0, 1, 2,...9, and if z_3 is any integer $>$, $=$, or <0, then the series $\sum_{n=0}^{\infty} \frac{z_n}{10^n}$ is convergent.

We know that the series $\Sigma \frac{1}{10^n}$ is convergent ; it follows from Art. 7·32 that $\sum_0^\infty \frac{z_n}{10^n}$ is also convergent.

Thus we see that an infinite decimal fraction may also be regarded as an infinite series. In this case, we may say that every infinite decimal fraction is convergent and, therefore, represents a definite real number.

Comparison Tests for Convergence of Positive Term Series.

7·33. *If the series Σu_n is convergent, and the series Σv_n has the property $0\leqslant v_n\leqslant u_n$ for all values of n, or for all values of $n>m$, then the series Σv_n is also convergent.*

Let $U_n=\sum_1^n u_n$ and $V_n=\sum_1^n v_n$.

If $v_n \leqslant u_n$ for $n > m$, we have

$$v_{m+1}+v_{m+2}+\ldots+v_n \leqslant u_{m+1}+u_{m+2}+\ldots+u_n < U,$$

where $\sum_1^\infty u_n = U.$

Thus $$V_n < V_m + U,$$

so that V_n is a bounded increasing sequence.

$\therefore$ Σv_n is convergent.

Similarly we prove the theorem :—

If Σu_n is divergent, and $v_n > u_n$ for all values of $n > m$, then Σv_n is also divergent.

7·34. *Σu_n and Σv_n are two positive-term series. If $a \leqslant \frac{u_n}{v_n} < b$, for all values of $n > m$, where a and b are positive constants, or if $\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = k \neq 0$, the two series Σu_n and Σv_n converge or diverge simultaneously.*

Further, if $\text{Lt}\ \frac{u_n}{v_n} = 0$ *and* Σv_n *converges, then* Σu_n *also converges ; and if* $\text{Lt}\ \frac{u_n}{v_n} \to \infty$ *and* Σv_n *diverges,* Σu_n *also diverges.*

The case $a \leqslant \frac{u_n}{v_n} \leqslant b$ follows at once from Art. 7·33.

Again, when $\text{Lt}\ \frac{u_n}{v_n} = k$, $\epsilon > 0$; $\exists m.\ k-\epsilon < \frac{u_n}{v_n} < k+\epsilon$, whenever $n > m$.

$\therefore$ $u_n < (k+\epsilon)v_n$ for all $n > m$.

$\therefore$ By (i), if Σv_n is convergent, Σu_n is also convergent.

Again, $u_n > (k-\epsilon)v_n$, for all values of $n > m$;

$\therefore$ if Σv_n is divergent, Σu_n is also divergent.

Since $\text{Lt}\ \frac{v_n}{u_n} = \frac{1}{k}$, we can similarly prove the convergence or divergence of Σv_n, when Σu_n converges or diverges.

Further, if $\text{Lt}\ \frac{u_n}{v_n} = 0$, $\frac{u_n}{v_n} < \epsilon$ for $n > m$,

i.e., $u_n < \epsilon v_n$, $\therefore$ if Σv_n converges, Σu_n also converges.

But, if $\text{Lt}\ \frac{u_n}{v_n} \to \infty$, we can find m such that $\frac{u_n}{v_n} > G$, for $n > m$, however large G may be.

$$\therefore\ u_n > G v_n.$$

Hence, if Σv_n is divergent, then Σu_n is also divergent.

Note 1. When $\text{Lt}\ \frac{u_n}{v_n}=k$ (a constant), Σu_n behaves at ∞ as Σv_n does.

Def.—The function $\phi\ (n)$ is said to behave at ∞ like the function $\psi\ (n)$, if $\underset{n\to\infty}{\text{Lt}}\ \phi(n)/\psi(n)=$a finite number, different from zero, and is written as $\phi\ (n) \sim \psi(n)$.

Note 2. This test is *sufficient* only and is not *necessary*.

For example, if $u_n=\frac{1}{n2^{n-2}}$ and $v_n=\frac{1}{2^{n-1}}$,

then $\text{Lt}\ \frac{v_n}{u_n}=\text{Lt}\ \frac{n}{2}\to\infty$, and $\text{Lt}\ \frac{u_n}{v_n}=0$. But here Σu_n and Σv_n are both convergent.

7·35. *For all values of $n > m$,*

if $\frac{v_{n+1}}{v_n}<\frac{u_{n+1}}{u_n}$, and Σu_n is convergent, then Σv_n is also convergent;

and if $\frac{v_{n+1}}{v_n}>\frac{u_{n+1}}{u_n}$ and Σu_n is divergent, then Σv_n is also divergent.

We have, if $\frac{v_{n+1}}{v_n}<\frac{u_{n+1}}{u_n}$,

$$v_{m+1}+v_{m+2}+\ldots=v_{m+1}\left\{1+\frac{v_{m+2}}{v_{m+1}}+\frac{v_{m+3}}{v_{m+2}}\cdot\frac{v_{m+2}}{v_{m+1}}+\ldots\right\}$$

$$<v_{m+1}\left\{1+\frac{u_{m+2}}{u_{m+1}}+\frac{u_{m+3}}{u_{m+2}}\cdot\frac{u_{m+2}}{u_{m+1}}+\ldots\right\}$$

$$<\frac{v_{m+1}}{u_{m+1}}\left\{u_{m+1}+u_{m+2}+u_{m+3}+\ldots\right\}.$$

Since Σu_n converges to U, say,

$V_n<\frac{v_{m+1}}{u_{m+1}}U+V_m$, where $V_n=v_1+v_2+\ldots+v_n$.

Thus V_n is a bounded increasing sequence. Hence the v-series also converges.

Similarly follows the second case, when Σu_n is divergent.

Or, $\because\ \frac{v_{n+1}}{u_{n+1}}<\frac{v_n}{u_n}<\frac{v_{n-1}}{u_{n-1}}<\ldots<\frac{v_m}{u_m}$,

we may apply 7·34, [since m is finite, $\frac{v_m}{u_m}$ is also finite.]

In like manner, we can discuss the case for divergence.

7·36. *The series*

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\ldots \textit{to infinity}$$

converges if $p>1$,
diverges if $p\leqslant 1$.

and

Case I. Let $p>1$.

$$\frac{1}{1^p}=1$$

$$\frac{1}{2^p}+\frac{1}{3^p}<\frac{2}{2^p}=\frac{1}{2^{p-1}},$$

$$\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}<\frac{4}{4^p}=\frac{1}{4^{p-1}},$$

$$\frac{1}{8^p}+\frac{1}{9^p}+\frac{1}{10^p}+\ldots+\frac{1}{15^p}<\frac{8}{8^p}=\frac{1}{8^{p-1}},$$

....

Now adding m of these inequalities together, we get

$$S_k\equiv\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots+\frac{1}{k^p}<1+\frac{1}{2^{p-1}}+\frac{1}{2^{2p-2}}+\frac{1}{2^{3p-3}}+\ldots$$

to n terms

$$=\frac{1-r^m}{1-r},$$

where $r=1/2^{p-1}$ and $k=2^m-1$. Thus $k\to\infty$ with m.

Since $p>1$, $r=1/2^{p-1}<1$.

$\therefore$ S_k is evidently increasing and is bounded, and consequently S_k tends to a limit

$$S\leqslant\frac{1}{1-r},\quad r=\frac{1}{2^{p-1}}.$$

Thus the series converges if $p>1$.

Case II. Let $p=1$.

$$1=\tfrac{1}{2}+\tfrac{1}{2},$$
$$\tfrac{1}{2}=\quad\tfrac{1}{2},$$
$$\tfrac{1}{3}+\tfrac{1}{4}>\tfrac{2}{4}=\quad\tfrac{1}{2},$$
$$\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8}>\tfrac{4}{8}=\quad\tfrac{1}{2},$$
$$\tfrac{1}{9}+\tfrac{1}{10}+\ldots+\tfrac{1}{16}>\tfrac{8}{16}=\quad\tfrac{1}{2},$$
.....

Now add $m+1$ such inequalities. The number of terms of the series thus added would be

$$1+1+2+2^2+\ldots\ldots\text{to } (m+1)\text{ terms}=2^m=n, \text{ say.}$$

$$\therefore\ S_n=1+\tfrac{1}{2}+\ldots\ldots+\tfrac{1}{n}>\frac{m+1}{2},\ \text{where } n=2^m.$$

Now given G, however large, we can find m_0, such that

$$\frac{m+1}{2} > G \quad \text{for all } m > m_0.$$

$$\therefore\ S_n > G,\ n > 2^{m_0}.$$

Thus the series diverges.

Case III. Let $p<1$; $\dfrac{1}{n^p} > \dfrac{1}{n}$.

Thus every term of the series $\Sigma\dfrac{1}{n^p}$ (after the first) is greater than the corresponding term of the known divergent series $\Sigma\dfrac{1}{n}$.

Thus $\Sigma\dfrac{1}{n^p}$ diverges when $p<1$.

Examples VII (*c*)

1. Test the convergence of the series

$$\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\frac{7}{4.5.6}+\ldots\ldots$$

[Here $u_n=\dfrac{2n-1}{n(n+1)(n+2)}$. The numerator is of the first degree and the denominator of the third degree in n. This suggests that the given series may be compared with $\Sigma\dfrac{1}{n^2}$.

If $v_n=\dfrac{1}{n^2}$,

$$\frac{u_n}{v_n}=\frac{(2n-1)n^2}{n(n+1)(n+2)}=\frac{\left(2-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} \to 2, \text{ as } n\to\infty.$$

Hence Σu_n and Σv_n converge or diverge together.

But $\Sigma\dfrac{1}{n^2}$ converges, so does, therefore Σu_n.]

2. Show that the series

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots \qquad \text{is convergent.}$$

[Here $u_n=\dfrac{1}{n(n+1)}$. We can compare it with $\dfrac{1}{n^2}$.]

3. Show that the series $\Sigma\dfrac{1}{n^2+1}$ converges.

4. Show that the series

$$1+\frac{1}{2.2^{1+\frac{1}{100}}}+\frac{1}{3.3^{1+\frac{1}{100}}}+\frac{1}{4.4^{1+\frac{1}{100}}}+\ldots$$

is convergent.

5. Show that the series

$$1+\frac{2^{1+\frac{1}{100}}}{2}+\frac{3^{1+\frac{1}{100}}}{3}+\ldots$$

is divergent.

6. Show that the series

$$\frac{1.2}{3^2.4^2}+\frac{3.4}{5^2.6^2}+\frac{5.6}{7^2.8^2}+\ldots\ldots$$

converges.

7. Show that the series $\Sigma\frac{1}{\sqrt{n}+\sqrt{n+1}}$ is divergent.

8. Prove that the series $\Sigma\frac{1}{\sqrt{n(n+1}}$ diverges.

Investigate the behaviour of the series

9. $\Sigma\frac{1}{(an+b)^{1+a}}$ 10. $\Sigma\frac{1}{(2n+1)^p}$.

11. Show that $\Sigma\frac{1+n}{1+n^2}$ is divergent.

12. Show that the series $\Sigma\frac{n}{n^2+n-1}$ diverges.

[For large values of n the dominant term in the denominator is n^2. This suggests that $\frac{n}{n^2+n-1}$ may be compared with $\frac{n}{n^2}$, *i.e.* $\frac{1}{n}$]

13. Show that the series $\Sigma\frac{n+\sqrt{n}}{n^2-n}$ diverges.

Investigate the behaviour of the following series :—

14. $\Sigma\sin\frac{1}{n}$. [when $n\to\infty$, $\frac{1}{n}\to 0$, so that $\lim_{n\to\infty}(\sin\frac{1}{n}/\frac{1}{n})=1$.

$\therefore$ $\Sigma\sin\frac{1}{n}$ as $\Sigma\frac{1}{n}$ converge or diverge together. But $\Sigma\frac{1}{n}$ diverges; consequently $\Sigma\sin\frac{1}{n}$ also diverges.]

15. $\Sigma\frac{1}{n}\sin\frac{1}{n}$. 16. $\Sigma\sin^2\frac{1}{n}$.

17. $\Sigma\tan^{-1}\frac{1}{n}$. [Hint. $\tan^{-1}\frac{1}{n}\sim\frac{1}{n}$ as $n\to\infty$.]

18. $\Sigma \frac{1}{\sqrt{n}} \tan \frac{1}{n}$. [Hint. $\tan \frac{1}{n} \sim \frac{1}{n}$ as $n \to \infty$.] 19. $\Sigma \cot^{-1} n$.

20. $\Sigma \frac{1}{n^{1+1/n}}$.

Let $u_n = \frac{1}{n^{1+\frac{1}{n}}}$. Take $v_n = \frac{1}{n}$, $\therefore \frac{u_n}{v_n} = \frac{1}{n^{\frac{1}{n}}}$.

To find the limit $n^{\frac{1}{n}}$ as $n \to \infty$, we note $n^{\frac{1}{n}}$ is a decreasing sequence, for

$$n^{\frac{1}{n}} \gtrless (n+1)^{\frac{1}{n+1}},$$

according as $$n^{n+1} \gtrless (n+1)^n,$$

i.e., $$n \gtrless \left(1+\frac{1}{n}\right)^n.$$

and this condition is satisfied when $n \geqslant 3$. Thus $n^{\frac{1}{n}}$ decreases when $n \geqslant 3$, and since $n^{\frac{1}{n}} > 1$, $n^{\frac{1}{n}}$ tends to a limit $l \geqslant 1$.

Suppose $l > 1$ and let $l = 1+h$;

then $$n^{\frac{1}{n}} > 1+\delta \qquad \text{where } h > \delta,$$

or $$n > (1+\delta)^n$$

$$> 1+n\delta + \frac{n(n-1)}{2}\delta^2,$$

or $$> n\delta + \frac{n(n-1)}{2}\delta^2.$$

$$\therefore \quad n(1-\delta) > \frac{n(n-1)}{2}\delta^2,$$

or $$1-\delta > \frac{n}{2}\delta^2, \text{ which is untrue for } \delta \text{ being}$$

constant, since $\frac{n}{2}\delta^2$ can be made as large as we please.

Hence $$\operatorname*{Lt}_{n \to \infty} n^{\frac{1}{n}} = 1.$$

It follows that $$\operatorname*{Lt}_{n \to \infty} \frac{u_n}{v_n} = 1,$$

and since $\Sigma \frac{1}{n}$ diverges, so does the other series.

21. Show that the series $\Sigma\{(n^3+1)^{\frac{1}{3}}-n\}$ is convergent.

[Hint. $u_n=\{(n^3+1)^{\frac{1}{3}}-n\}=n\left\{\left(1+\frac{1}{n^3}\right)^{\frac{1}{3}}-1\right\}$

$$=n\left\{\left(1+\frac{1}{3n^3}+\cdots\right)-1\right\}=\frac{1}{3n^2}+\cdots\sim\frac{1}{n^2}.$$

22. Show that the series $\Sigma\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)$ is convergent.

23. Show that the series $\Sigma\frac{\sqrt{n}}{n^2+1}$ is convergent.

24. Show that the series $1+\frac{1}{4^{\frac{2}{3}}}+\frac{1}{9^{\frac{2}{3}}}+\frac{1}{16^{\frac{2}{3}}}+\cdots$ is convergent.

25. Investigate the behaviour of the series

$$\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\frac{5}{4^p}+\cdots$$

26. Show that the series $\Sigma\left(\frac{1}{n}-\log\frac{n+1}{n}\right)$ is convergent.

[Hint. It behaves as $\Sigma\frac{1}{n^2}$.]

27. Show that the series $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\cdots$ is divergent.

[Hint. $u_{n+1}=\frac{n^n}{(n+1)^{n+1}}=\frac{1}{\left(1+\frac{1}{n}\right)^n(n+1)}\sim\frac{1}{e(n+1)}$.]

28. Examine the series $\frac{2^p}{1^q}+\frac{3^p}{2^q}+\frac{4^p}{3^q}+\cdots$,

p and q being positive numbers.

29. Test the convergence of the series

$$1+\frac{2}{2^2+1}+\frac{3}{3^2+1}+\frac{4}{4^2+1}+\cdots$$

30. Show that the series

$\Sigma\frac{n^p}{(n+1)^{p+q}}$ converges if $q>1$, and diverges if $q<1$.

31. Test the convergence of the series

$$\frac{2^3}{1^k+3^k}+\frac{3^3}{2^k+4^k}+\frac{4^3}{3^k+5^k}+\cdots \qquad (P.U.\ 1939)$$

32. If Σu_n is convergent, show that so are

$$\Sigma u_n^2 \text{ and } \Sigma\frac{u_n}{1-u_n},\ u_n\neq 1.$$

[$\because$ Σu_n converges, we have $u_n < \frac{1}{2}$ for $n > n_0$. Then for $n > n_0$, $u_n^2 \leqslant u_n$, or $\frac{u_n}{1-u_n} \leqslant 2u_n$. The result now follows from Comparison tests.]

33. If Σu_n^2, Σv_n^2 are convergent, so is $\Sigma u_n v_n$.

[$\because$ Σu_n^2, Σv_n^2 are convergent, so is $\Sigma (u_n^2 + v_n^2)$. Now $u_n v_n < \frac{1}{2}(u_n^2 + v_n^2)$. The result follows from Comparison tests.]

7·37 The Comparison tests can only be useful if the student is already acquainted with a large number of convergent and divergent series with positive terms. He should, therefore, lay in as large a stock as possible of the series whose convergence or divergence is known. But as the convergence of a series is of vital importance, it is desirable also to have tests of an independent nature for the convergence or divergence of a given series. We now proceed to deduce such tests.

7·4. D' Alembert's Ratio Test.

A positive-term series Σu_n

converges if $$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = r < 1,$$

and diverges if $$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = r > 1.$$

To prove the first statement choose a number $r+\epsilon$ such that $r < r+\epsilon < 1$.

Since $$\operatorname{Lt}_{n \to \infty} \frac{u_{n+1}}{u_n} = r,$$

$\epsilon > 0$: $\exists m$. $\frac{u_{n+1}}{u_n} < r+\epsilon$, whenever $n \geqslant m$.

$$\therefore \quad u_m + u_{m+1} + \ldots u_{m+p} = u_m \left(1 + \frac{u_{m+1}}{u_m} + \frac{u_{m+2}}{u_{m+1}} \cdot \frac{u_{m+1}}{u_m} + \ldots + \frac{u_{m+p}}{u_{m+p-1}} \ldots \frac{u_{m+1}}{u_m} \right)$$

$$< u_m [1 + (r+\epsilon) + (r+\epsilon)^2 + \ldots\ldots\ldots + (r+\epsilon)^p]$$

$$< u_m \frac{1}{1-r-\epsilon}, \qquad \because \quad r+\epsilon < 1.$$

$$\therefore \quad U_{m+p} < U_{m-1} + \frac{u_m}{1-r-\epsilon}.$$

As $U_{n+1} > U_m$ for $n > m$, U_{m+p} is an increasing sequence and is bounded above.

Thus the sequence (U_{m+p}) tends to a limit U, as $p \to \infty$.

The second statement follows from similar considerations.

7·5. Cauchy's Radical Test or Root Test.

A positive-term series Σu_n *converges if* $\underset{n\to\infty}{\text{Lt}}\ u_n^{\frac{1}{n}} = r < 1$, and diverges, if $\text{Lt}\ u_n^{\frac{1}{n}} = r > 1$.

1st case. For, $\text{Lt}\ u_n^{\frac{1}{n}} = r$, Now $r < 1$; hence we may choose ϵ positive and sufficiently small so that $r + \epsilon < 1$.

Now, since $\text{Lt}\ u_n^{\frac{1}{n}} = r$, it is possible to find m so large that

$u_n^{\frac{1}{n}} < r + \epsilon$ for $n \geqslant m$,

$\therefore u_n < (r+\epsilon)^n$ for $n \geqslant m$.

Hence from and after the mth term, the terms of the series Σu_n are less than those of the convergent series $\Sigma(r+\epsilon)^n$.

$\therefore$ Σu_n is convergent.

2nd case. On the hypothesis in this case, the geometric series is divergent, since $u_n > (r-\epsilon)^n$.

Note 1. The Root and Ratio Tests are by far the most important tests used in practice. In some books they are stated in the form:

A positive term series Σu_n *converges if for every* n, $\sqrt{u_n} < 1$, or $u_{n+1}/u_n > 1$.

But, in both cases, it is essential for convergence that $\sqrt[n]{u_n}$ and u_{n+1}/u_n, respectively, should be *ultimately less than a fixed proper fraction*, r, and it does not at all suffice for convergence that $\sqrt[n]{u_n} < 1$, or $\dfrac{u_{n+1}}{u_n} < 1$ for every n. For $\Sigma\dfrac{1}{n}$ which is divergent, we always have $\sqrt[n]{\dfrac{1}{n}} < 1$.

and also
$$\frac{1}{n+1} - \frac{1}{n} = 1 - \frac{1}{n+1} < 1.$$

Note 2. If $\text{Lt}\dfrac{u_{n+1}}{u_n} = 1$, or $\text{Lt}\sqrt[n]{u_n} = 1$, no information relative to the convergence or divergence is furnished by either test. For, if either limit $= 1$, it will not be possible to find a number k such that $\dfrac{u_n}{u_{n+1}} - 1 > k > 0$ for $n \geqslant m$.

Examples VII (*d*)

1. Determine whether the series whose general term is

$$\frac{\sqrt{n}}{\sqrt{n^2+1}}x^n,$$

converges or diverges.

Let $$u_n=\frac{\sqrt{n}}{\sqrt{n^2+1}}x^n,$$

Then $$\frac{u_{n+1}}{u_n}=\frac{(n+1)^{\frac{1}{2}}}{[(n+1)^2+1]^{\frac{1}{2}}}x^{n+1}.\frac{\sqrt{n^2+1}}{\sqrt{n}}\frac{1}{x^n}$$

$$=\frac{\left(1+\frac{1}{n}\right)^{\frac{1}{2}}\left(1+\frac{1}{n^2}\right)^{\frac{1}{2}}}{\left(1+\frac{2}{n}+\frac{2}{n^2}\right)^{\frac{1}{2}}}x\to x, \text{ as } n\to\infty.$$

$\therefore$ The series converges if $x<1$, and diverges if $x>1$.

When $x=1$, $$u_n=\frac{\sqrt{n}}{\sqrt[n]{1+\frac{1}{n^2}}\ \sqrt{n}\sqrt{1+\frac{1}{n^2}}},$$

$$\therefore \quad \frac{u_n}{1/\sqrt{n}}=\frac{1}{\sqrt{1+\frac{1}{n^2}}}\to 1, \text{ as } n\to\infty.$$

$\therefore$ $\Sigma\frac{\sqrt{n}}{\sqrt{n^2+1}}$ and $\Sigma\frac{1}{\sqrt{n}}$ behave alike.

But since $\Sigma\frac{1}{\sqrt{n}}$ is divergent, so is $\Sigma\frac{\sqrt{n}}{\sqrt{n^2+1}}$.

Thus the given series converges if $x<1$, and diverges if $x\geqslant 1$.

2. Determine whether the series whose general term is given by

$$\left[1+\left(\frac{1}{\sqrt{n}}\right)\right]^{-n^{3/2}},$$

is convergent or divergent. (*P. U. 1941*)

Let $$u_n=\frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{n^{3/2}}},$$

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Then $$u_n^{\frac{1}{n}} = \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} \to \frac{1}{e}, \text{ as } n\to\infty ;$$

and since $\frac{1}{e} < 1$, the series converges.

3. The series

$$\Sigma \frac{\{(n+1)r\}^n}{n^{n+1}}$$

is convergent if $r<1$, and divergent if $r \geqslant 1$. [*Math. Trip. 1928*]

Let $$u_n = \frac{\{(n+1)r\}^n}{n^{n+1}}.$$

Now $$u_n^{1/n} = \frac{n+1}{n^{1+\frac{1}{n}}} r = \frac{1+\frac{1}{n}}{n^{1/n}} r$$

$$\to r, \text{ as } n\to\infty .$$

[Limit of $n^{\frac{1}{n}}$ has been proved equal to 1 ; see page 201].

$\therefore$ Σu_n converges if $r<1$, and diverges if $r>1$.

If $r=1$, $$u_n = \frac{(n+1)^n}{n^{n+1}} = \left(1+\frac{1}{n}\right)^n \cdot \frac{1}{n}.$$

$$\therefore \quad \frac{u_n}{1/n} \to e.$$

Now $\Sigma \frac{1}{n}$ is divergent, hence Σu_n also is divergent.

Investigate the behaviour of the series whose general terms are

4. $\frac{2^n\,(n\,!)}{n^n}$. 5. $\frac{3^n\,(n\,!)}{n^n}$. 6. $\sqrt[n]{\frac{n\,!}{n^n}}$. 7. $\sqrt{\frac{5^n-1}{6^n-1}}$.

8. Prove that the series

$$\frac{1}{2}+\frac{1}{2.2^2}+\frac{1}{3.2^3}+\frac{1}{4.2^4}+\ldots$$

converges.

9. Show that the series $\Sigma n^{-3/2}$ and $\Sigma n^{-11/10}$ are convergent, and $\Sigma n^{-1/2}$ and $\Sigma n^{-10/11}$ are divergent.

10. Show that the series

$$1+\frac{x}{2^2}+\frac{x^2}{3^2}+\frac{x^3}{4^2}+\ldots$$

converges.

11. The series $\Sigma\frac{r^n}{n!}$ and $\Sigma\frac{r^n}{n^n}$ are convergent for all r; and $\Sigma\, n!\, r^n$ and $\Sigma n^n r^n$ for no r, except $r=0$. [*Math. Trip. 1935, 36*]

12. Show that the series $\Sigma\frac{n^a}{n!}$ converges for all finite values of a.

13. Show that the series

Sol. $$\left(\frac{1}{3}\right)^2+\left(\frac{1.2}{3.5}\right)^2+\left(\frac{1.2.3}{3.5.7}\right)^2+\ldots$$

converges.

14. Show that the series $\Sigma\frac{x^n}{1+x^{2n}}$ converges for $x\gtrless 1$, and diverges for $x=1$.

15. Show that the series $\Sigma\frac{n^2-1}{n^2+1}x^n$ converges for $x<1$, and diverges for $x\geqslant 1$.

Sol. $$u_n=\frac{n^2-1}{n^2+1}x^n.$$

$$\therefore\quad \frac{u_{n+1}}{u_n}=\frac{(n+1)^2-1}{(n+1)^2+1}\cdot\frac{n^2+1}{n^2-1}x=\frac{(n^2+2n)(n^2+1)}{(n^2+2n+2)(n^2-1)}x$$

$$=\frac{\left(1+\frac{2}{n}\right)\left(1+\frac{1}{n^2}\right)}{\left(1+\frac{2}{n}+\frac{2}{n^2}\right)\left(1-\frac{1}{n^2}\right)}x$$

$$\to x, \text{ when } n\to\infty.$$

$\therefore$ The series converges if $x<1$, and diverges if $x>1$.

When $x=1$, $$u_n=\frac{n^2-1}{n^2+1}=\frac{1-\frac{1}{n^2}}{1+\frac{1}{n^2}}\to 1, \text{ as } n\to\infty.$$

Thus u_n does not tend to zero; the series, therefore, diverges.
Thus the series converges if $x<1$, and diverges if $x\geqslant 1$.

16. Show that the series $\Sigma\frac{2^n-2}{2^n+1}x^n$ converges for $x<1$, and diverges for $x\geqslant 1$.

17. Test the convergence of the series

$$1+\frac{x^2}{2^p}+\frac{x^4}{4^p}+\frac{x^6}{6^p}+\ldots$$

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18. Discuss the convergence of the series

$$\sum \frac{x^n}{n^{\frac{1}{2}}+a}.$$

19. Discuss the convergence of the series

$$\sum \frac{1.2\ldots n}{4.7\ldots(3n+1)}x^n\,; \qquad \sum \frac{1.2\ldots n}{7.10\ldots(3n+4)}.$$

20. Show that the series $\sum \frac{(n+1)(n+2)\ldots(n+n)}{n^n}$ diverges.

21. Discuss the convergence of

$1+a+b^2+a^3+b^4+a^5+b^6+\ldots$, if $0<a<b<1$.

Sol. $\frac{u_{2n+1}}{u_{2n}}=\frac{b^{2n}}{a^{2n-1}}=a\left(\frac{b}{a}\right)^{2n}\to\infty$, as $n\to\infty$.

$$\frac{u_{2n+2}}{u_{2n+1}}=b\left(\frac{a}{b}\right)^{2n+1}\to 0, \text{ as } n\to\infty.$$

The ratio test fails as $\text{Lt}\frac{u_{n+1}}{u_n}$ fluctuates between the limits which include unity.

$$u_{2n}^{\frac{1}{2n}}=(a^{2n-1})^{\frac{1}{2n}}=a^{1-\frac{1}{2n}}\to a<1, \text{ as } n\to\infty,$$

and $u_{2n+1}^{\frac{1}{2n+1}}=b^{\frac{2n}{2n+1}}\to b<1$, as $n\to\infty$.

The test shows that the series converges.

This example shows that the root-test is more general than the ratio test, for the former can be applied when the latter is applicable and also sometimes when the latter cannot be applied.

22. Show that the series

$$1+\frac{a}{2}+\frac{a}{3}+\frac{a^2}{4}+\frac{a^2}{5}+\frac{a^3}{6}+\frac{a^3}{7}\ldots\ldots\ldots$$

is convergent if $0<a<1$.

[*Cf.* with the convergent series $1+a+a+a^2+a^2+\ldots$ for the result.]

The ratio test fails. The root-test, however, gives the information.]

7·51. We give below two more tests due to Cauchy, which are applicable to the series whose terms form a monotonic sequence.

I. Cauchy's Condensation Test.

If $\sum_{n=1}^{\infty} \phi(n)$ *is a series whose terms are positive and steadily decreasing, as* n *increases, then it converges or diverges with the series* $\sum_{n=1}^{\infty} a^n\phi(a^n)$, *where* a *is a positive integer* > 1.

Since $\phi(n) \leqslant \phi(n-1)$,

$\therefore \quad \phi(a^k+1)+\phi(a^k+2)+\phi(a^k+3)+\ldots+\phi(a^{k+1})$

$$< (a-1)a^k\phi(a^k),$$

and

$$> \frac{a-1}{a}a^{k+1}\phi(a^{k+1}),$$

for, the number of terms $= a^{k+1}-a^k = (a-1)a^k = \frac{a-1}{a}a^{k+1}$.

By giving to k values 0, 1, 2, 3,...we have

$$\phi(2)+\phi(3)+\phi(4)+\ldots+\phi(a) < (a-1)a^0\phi(1),$$

$$> \frac{a-1}{a}a\,\phi(a);$$

$$\phi(a+1)+\phi(a+2)+\ldots+\phi(a^2) < (a-1)a\phi(a),$$

$$> \frac{a-1}{a}a^2\phi(a^2),$$

.....

$\therefore$ adding up, we get

$$\phi(2)+\phi(3)+\phi(4)+\ldots\phi(a^{k+1})$$

$$< (a-1)[a^0\phi(1)+a\phi(a)+a^2\phi(a^2)+\ldots+a^k\phi(a^k)]$$

and

$$> \frac{a-1}{a}\left\{a\phi(a)+a^2\phi(a^2)+\ldots+a^{k+1}\phi(a^{k+1})\right\}.$$

Hence the theorem.

Ex. 1. Consider the series $\Sigma\frac{1}{n}$. Here $\phi(n)=\frac{1}{n}$.

Take $a=2$, $\Sigma\frac{1}{n}$ certainly diverges, for it converges, or diverges with the series $\Sigma 2^n\phi(2^n)=\Sigma\frac{2^n}{2^n}=1+1+1+\ldots$, which is divergent.

Ex. 2. Consider the series $\Sigma\frac{1}{n^p}$. Here $\phi(n)=\frac{1}{n^p}$.

$\therefore\ \Sigma 2^n.\frac{1}{(2^n)^p}=\Sigma\left\{\frac{1}{2^{p-1}}\right\}^n$, which is a geometric series.

The given series, therefore, converges, or diverges, according as $p>1$, or $p\leqslant 1$.

Note. It is called the Condensation test, for a small proportion of all the terms of the series is sufficient to establish the convergence or divergence of the whole series. The convergence or divergence of $\Sigma a^n\phi(a^n)$ is often more easily determined than that of the series $\Sigma\phi(n)$ itself. It is just in this that the value of the test lies.

7·52. Cauchy's Integral Test.

If $\phi(x)$ is a function of x, which is positive and continuous for all values of $x>1$, and never increases with x, the series

$$\phi(1)+\phi(2)+\phi(3)+\dots$$

converges or diverges with the integral $\int_1^\infty \phi(x)\,dx$.

Let $$f(x)=\int_1^x \phi(x)\,dx.$$

The series whose general term is $f(n+1)-f(n)$ converges or diverges with the integral $\operatorname{Lt}_{n\to\infty} \int_1^{n+1} \phi(x)\,dx$.

But by the Mean-value Theorem,

$$f(n+1)-f(n)=\phi(n+\theta),$$

where $|\theta|<1$. But, since

$$\phi(n) > \phi(n+\theta) > \phi(n+1),$$

$$\therefore \quad \sum_1^n \phi(n) > \sum_1^n \phi(n+\theta) > \sum_1^n \phi(n+1).$$

Thus the series $\Sigma\,\phi(n+\theta)$ converges if $\sum_1^n \phi(n)$ converges, and diverges if $\sum_1^n \phi(n+1)$ diverges. Similarly $\Sigma\phi(n+1)$ converges if $\Sigma\,\phi(n+\theta)$ converges, and $\sum_1^n \phi(n)$ diverges if $\Sigma\phi(n+\theta)$ diverges. Hence the theorem.

Ex. Test for convergence or divergence the series

(*i*) $\sum_1^\infty \frac{1}{n^p}$ (*ii*) $\sum_{n=2}^\infty \frac{1}{n(\log n)^p}$.

Now for (*i*), since $\phi(n)=u_n=\frac{1}{n^p}$, $\quad\therefore\quad \phi(x)=\frac{1}{x^p}$.

If $p\geqslant 0$, $\phi(x)$ decreases as x increases, and we can apply the test.

Hence $\int_1^s \phi(x)\,dx=\int_1^s \frac{dx}{x^p}=\frac{1}{1-p}(x^{1-p}-1)$, or $\log x$,

according as $p \gtrless 1$, or $p=1$.

Thus when $n\to\infty$, the Integral converges only if $p>1$. Hence the series converges if $p>1$, and diverges if $p\leqslant 1$.

When $p=1$, the integral shows that the harmonic series $\sum_1^\infty \frac{1}{n}$ is divergent.

We infer also that $\operatorname{Lt}_{n\to\infty}\left\{1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\log n\right\}$ exists, and lies between 0 and 1.

This limit is called *Euler's constant.*

(*ii*) Here $\phi(x)=\dfrac{1}{x(\log x)^p}$,

and so $$\int_1^x \phi(x)\,dx = \frac{1}{p-1}\left\{(\log x)^{1-p}-(\log 2)^{1-p}\right\},$$

or $\log(\log x/\log 2)$, according as $p \gtrless 1$, or $p=1$.

Thus the series converges if $p>1$, and diverges if $p\leqslant 1$.

Note 1.—Similarly we can show that the series

$$\sum_{n=m}^\infty \frac{1}{n\log n(\log\log n)^p},$$

$$\sum_{n=m}^\infty \frac{1}{n\log n\log\log n(\log\log\log n)^p},$$

etc., converge if $p>1$, and diverge if $p\leqslant 1$, m being so chosen that all the terms of each series are positive.

Note 2.—The convergence of the series $\Sigma\dfrac{1}{(n\log n)^p}$ may also be deduced from the Condensation Test.

The series converges or diverges with the series

$\Sigma\dfrac{2^n}{2^n(\log 2^n)^p}$, *i.e.*, with $\Sigma\dfrac{1}{(n\log 2)^p}$, *i.e.*, with $\Sigma\dfrac{1}{n^p(\log 2)^p}$.

Now $\dfrac{1}{(\log 2)^p}$ is a constant. Therefore, the given series converges or diverges with $\Sigma\dfrac{1}{n^p}$. Similarly, we may establish the results in other cases.

7·53. As Cauchy's Integral Test is not easy to apply except in simple cases and D'Alembert's Tests and Cauchy's Tests fail when $\operatorname{Lt}\dfrac{u_{n+1}}{u_n}=1$, and $\operatorname{Lt} u_n^{\frac{1}{n}}=1$, other sharper, tests for convergence or divergence are sometimes needed. We give below two such tests.

7·6. Raabe's Test. *In the positive-term series,* Σu_n,

if $\underset{n\to\infty}{\text{Lt}} \quad n\left(\frac{u_n}{u_{n+1}}-1\right)=r.$

the series converges if $r>1$,
and diverges if $r<1$.

Proof. Since $\underset{n\to\infty}{\text{Let}}\ n\left(\frac{u_n}{u_{n+1}}-1\right)=r,$

given $\epsilon>0$, there exists a value m of n such that

$$r-\epsilon<n\left(\frac{u_n}{u_{n+1}}-1\right)<r+\epsilon, \text{ when } n\geqslant m.$$

Hence $n\,u_n-(n+1)u_{n+1} > (r-1-\epsilon)u_{n+1}$, for $n > m$.
Giving to n the values, m, $m+1$, $m+2$, ...$m+p$,

$$m\,u_m-(m+1)u_{m+1} > (r-1-\epsilon)\,u_{m+1},$$
$$(m+1)u_{m+1}-(m+2)u_{m+2} > (r-1-\epsilon)u_{m+2},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(m+p-1)u_{m+p-1}-(m+p)u_{m+p} > (r-1-\epsilon)u_{m+p}.$$

Adding these inequalities,

$$m\,u_m-(m+p)u_{m+p} > (r-1-\epsilon)(s_{m+p}-s_m),$$

where $s_m=u_1+u_2+\ldots\ldots+u_m$, $s_{m+p}=u_1+u_2+\ldots\ldots+u_{m+p}$;

whence
$$s_{m+p}-s_m < \frac{1}{r-1-\epsilon}[m\,u_m-(m+p)u_{m+p}],$$
$$< \frac{1}{r-1-\epsilon}\,m\,u_m\,,$$
$$< \frac{1}{r-1-\epsilon}\,m\,u_m+s_m.$$

or
$$s_{m+p} < \frac{1}{r-1-\epsilon}\,m\,u_m+s_m.$$

Thus s_{m+p} is bounded for all values of p and is obviously an increasing function of p. Therefore s_{m+p} has a finite limit when $p\to\infty$.

$\therefore$ the series converges.

If $r<1$

$$n\left(\frac{u_n}{u_{n+1}}-1\right) < r+\epsilon, \text{ for } n\geqslant m.$$

$\therefore\ n\,u_n-(n+1)u_{n+1} < (r+\epsilon-1)u_{n+1}$, for $n\geqslant m$,
< 0, for, $r+\epsilon-1 < 0$. $n\geqslant m$

$\therefore\quad n\,u_n < (n+1)\,u_{n+1}.$

Now
$$m\,u_m < m+1)\,u_{m+1}$$
$$< (m+2)\,u_{m+2}$$
$$\cdots\cdots\cdots\cdots$$
$$< (m+p)u_{m+p}$$

$$\therefore\quad u_{m+p} > \frac{m\,u_m}{m+p},$$

$$\text{or} \qquad u_n > \frac{k}{n}, \quad \text{where } k = m\,u_m ;$$

so that the terms of Σu_n for $n > m$ are greater than those of the divergent series $k\,\Sigma 1/n$. Thus Σu_n is divergent.

Alternative Proof of Raabe's Test.

Since $\underset{n\to\infty}{\text{Lt}} \; n\left(\frac{u_n}{u_{n+1}} - 1\right) = r,$

$$\epsilon > 0 ; \; \exists\, n_0. \qquad r-\epsilon < n\left(\frac{u_n}{u_{n+1}} - 1\right) < r+\epsilon, \text{ for } n > n_0,$$

$$\text{or} \qquad 1+\frac{r-\epsilon}{n} < \frac{u_n}{u_{n+1}} < 1+\frac{r+\epsilon}{n}, \text{ for } n > n_0.$$

Let $v_n = \frac{1}{n^p},$

$$\therefore \quad \frac{v_n}{v_{n+1}} = \left(\frac{n+1}{n}\right)^p = \left(1+\frac{1}{n}\right)^p$$

$$= 1+\frac{p}{n}+\frac{p(p-1)}{2\,!\,n^2}\left(1+\frac{\theta}{n}\right)^{p-2} \qquad 0<\theta<1,$$

$$= 1+\frac{p}{n}+\frac{\lambda_n}{n^2}$$

where $\lambda_n = \frac{p(p-1)}{2\,!}\left(1+\frac{\theta}{n}\right)^{p-2}.$

The function λ_n of n remains less than a fixed number when $n\to\infty$.

Let $r > 1$. Choose p, $r-\epsilon > p > 1$, then, we can choose

$$1+\frac{r-\epsilon}{n} > 1+\frac{p}{n}+\frac{\lambda_n}{n^2}.$$

$$\therefore \quad \frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}, \qquad \text{or} \qquad \frac{u_{n+1}}{u_n} < \frac{v_{n+1}}{v_n}, \text{ when } n > n_0.$$

But, for $p > 1$, Σv_n converges, hence Σu_n also converges.

Suppose now that $r < 1$;
choose $r+\epsilon < 1$

$$\therefore \quad 1+\frac{r+\epsilon}{n} < 1+\frac{1}{n}.$$

Hence $$\frac{u_n}{u_{n+1}} < \frac{1+n}{n}, \qquad \text{when } u > n_0$$

or $$\frac{u_{n+1}}{u_n} > \frac{v_{n+1}}{v_n}, \qquad \text{where } v_n = \frac{1}{n}$$

Since $\Sigma \frac{1}{n}$ is divergent, so is, therefore, Σu_n

The tests may be stated as follows :

If u_n be a positive-term series, such that

$$\frac{u_{n+1}}{u_n}=\frac{1}{1+a_n},$$

and $\underset{n\to\infty}{Lt}\ na_n=r$, *the series*

converges if $r>1$,
diverges if $r<1$.

The test fails if $r=1$.

761. De Morgan's and Bertrand's Test. *In the positive-term series* Σu_n *if*

$$\underset{n\to\infty}{Lt}\ \log n\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}=r,$$

the series $\begin{cases}\text{converges} & \text{if } r>1,\\ \text{diverges} & \text{if } r<1.\end{cases}$

The proof depends upon the following lemma.

In the positive-term series Σu_n, *if*

$$\underset{n\to\infty}{Lt}\ \left[\frac{u_n}{u_{n+1}}\,n\log n-(n+1)\log\ (n+1)\right]=l,$$

the series $\begin{cases}\text{converges} & \text{if } l>0,\\ \text{diverges} & \text{if } l<0.\end{cases}$

Proof of the lemma,

Since the limit l is different from zero. we can find $\epsilon>0$, such that $l-\epsilon>0$, *e.g.*, $\epsilon=l/2$. Corresponding to this ϵ there exists a value m of n, such that

$$\frac{u_n}{u_{n+1}}\,n\log n-(n+1)\log\ (n+1)>(l-\epsilon)>0 \text{ for } n\geqslant m$$

or $u_n\,n\log n-(n+1)\log(n+1)\,u_{n+1}>(l-\epsilon)u_{n+1}$ for $n\geqslant m$;

putting $n=m, m+1,\ldots\ldots n$, and adding these inequalities we get

$$u_m\,m\log m-(n+1)\log(n+1)\,u_{n+1}>(l-\epsilon)(u_{m+1}+\ldots\ldots u_{n+1})$$
$$=(l-\epsilon)(s_{n+1}-s_m),$$

where $s_m=u_1+\ldots\ldots+u_m$, $s_{n+1}=u_1+u_2+\ldots\ldots+u_{n+1}$.

Hence $s_{n+1}-s_m<\dfrac{1}{l-\epsilon}[u_m\,m\log m-u_{n+1}(n+1)\log(n+1)]$

$$<\frac{1}{l-\epsilon}u_m\,m\log m,$$

or $$s_{n+1} < \frac{1}{l-\epsilon} u_m\, m \log m + s_m.$$

s_{n+1} is therefore bounded, and is obviously increasing.

Consequently s_{n+1} tends to finite limit. The series is therefore convergent.

In case l is negative,

$$u_n\, n \log n < u_{n+1}\,(n+1)\log(n+1) \qquad \text{for } n \geqslant m$$

$$\therefore\quad u_m.\, m \log m < u_{m+1}\,(m+1)\log(m+1)$$

$$< u_{m+2}\,(m+2)\log(m+2)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$< u_n.\, n \log n$$

$$\therefore\quad u_n > \frac{u_m.\, m \log m}{n \log n} = \frac{k}{n \log n},$$

so that the terms of the series Σu_n are, after the mth, greater than those of the divergent series $k \Sigma (n \log_n)^{-1}$.

Thus Σu_n is divergent.

Proof of the Theorem. Let

$$v_n = \frac{u_n}{u_{n+1}}\, n \log n - (n+1)\log(n+1) \text{ so that } \operatorname*{Lt}_{n\to\infty} v_n = l.$$

Now $$v_n = \log n \left[n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1 \right] - (n+1)\log\left(1+\frac{1}{n}\right)$$

$$= \log n \left[n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1 \right] - \left(1+\frac{1}{n}\right)\log\left(1+\frac{1}{n}\right)^n.$$

$$\therefore \operatorname*{Lt}_{n\to\infty} v_n = \operatorname*{Lt}_{n\to\infty} \log n \left[n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1 \right] - 1 = l,$$

Hence $$\operatorname*{Lt}_{n\to\infty} \log n \left[n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1 \right] = l+1,$$

The series converges if $l > 0$, and diverges if $l < 0$.

$\therefore$ The series Σu_n converges or diverges, according as

$$\operatorname*{Lt}_{n\to\infty} \log n \left[n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1 \right] = r,$$

is greater or less than unity.

7·8. Other Tests.

I. *In a positive-term series* Σu_n,

if $$\operatorname*{Lt}_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = r,$$

the series, Σu_n $\begin{cases} \textit{converges if } r>1, \\ \textit{diverges if } r<1. \end{cases}$

Comparing Σu_n with $\Sigma \frac{1}{n^p}$, $(p>1)$,

Σu_n converges, if $\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)^p$, for $n \geqslant m$,

i.e., if $$\log \frac{u_n}{u_{n+1}} > p \log\left(1+\frac{1}{n}\right)$$

$$> p\left[\frac{1}{n} - \frac{1}{2}\frac{1}{(\theta+n)^2}\right] \qquad 0<\theta<1$$

i.e., if $$\operatorname{Lt} n \log \frac{u_n}{u_{n+1}} > 1.$$

Similarly follows the second part.
This test fails when the limit is 1.

7·63. *In a positive-term series* Σu_n,

if $$\operatorname*{Lt}_{n\to\infty} \left(n \log \frac{u_n}{u_{n+1}} - 1\right) \log n = r,$$

the series $\begin{cases} \textit{converges if } r>1, \\ \textit{diverges if } r<1. \end{cases}$

Comparing Σu_n with $\frac{1}{n(\log n)^p}$ $(p>1)$,

Σu_n converges, if $\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)\left[\frac{\log(n+1)}{\log n}\right]^p$.

Let $$\operatorname*{Lt}_{n\to\infty} \left(n \log \frac{u_n}{u_{n+1}} - 1\right) \log n = r > 1,$$

$\epsilon>0$; $\exists m$, $\left(n \log \frac{u_n}{u_{n+1}} - 1\right) \log n > r-\epsilon > 1$ when $n \geqslant m$

or $$\log \frac{u_n}{u_{n+1}} > \frac{1}{n} + \frac{p}{n \log n} \quad \text{where } p = r-\epsilon, \text{ for } n \geqslant m$$

or $$\frac{u_n}{u_{n+1}} > e^{\frac{1}{n} + \frac{p}{n \log n}};$$

and by second Mean Value Theorem

$$= 1 + \frac{1}{n} + \frac{p}{n \log n} + \frac{\lambda_n}{n^2},$$

where $\lambda_n = \frac{1}{2}\left(1+\frac{p}{\log n}\right)^2 e^{\theta\left(\frac{1}{n}+\frac{p}{n\log n}\right)}$ $\qquad 0<\theta<1.$

λ_n is therefore bounded for values of n however large.

$$\therefore \quad \left[\left(\frac{u_n}{u_{n+1}}-1\right)n-1\right]\log n > p+\lambda_n.\frac{\log n}{n}$$

and $$\operatorname*{Lt}_{n\to\infty}\left[n\left(\frac{u_n}{u_{n+1}}-1\right)\right]\log n \geqslant p>1.$$

The result then follows by D' Morgan's Test.

In case $r<1$,

$$\left(n\log\frac{u_n}{u_{n+1}}-1\right)\log n<r+\epsilon<1.$$

The result then follows as in the above case ; and this is left as an exercise to the student.

7·64. *If Σu_n is a convergent series of positive and decreasing terms, prove that*

(*i*) $\operatorname*{Lt}_{n\to\infty} nu_n=0$, (*ii*) $\operatorname*{Lt}_{n\to\infty} n\log n.\,u_n=0$. (*Pringsheim*)

(i) If $\operatorname*{Lt}_{n\to\infty} nu_n=l\neq 0$, $\epsilon>0$; $\exists\ m.nu_n>l-\epsilon$, whenever $n\geqslant m$.

$$\therefore \quad u_n>\frac{l-\epsilon}{n}, \text{ when } n\geqslant m.$$

Hence Σu_n must diverge, which is a contradiction of hypothesis.

(*ii*) Suppose that $\operatorname*{Lt} n\log n.\,u_n=l\neq 0$.

$$\epsilon>0 ; \exists\ m.\ n\log n\,u_n\geqslant l-\epsilon, \text{ for } n\geqslant m,$$

or $$u_n>\frac{l-\epsilon}{n\log n}.$$

But $\sum_2^\infty \frac{1}{n\log n}$ is divergent, $\therefore \Sigma u_n$ diverges.

Hence $\operatorname*{Lt} n\log n.\,u_n=0$.

Note.—This theorem gives a sufficient test for divergence only. The converse of this theorem is not true. *The condition that u_n decreases as n increases, is essential.*

Ex. Test for convergence or divergence the series

$$x^2(\log 2)^q+x^3(\log 3)^q+x^4(\log 4)^q+\ldots \quad (1941\ B.\ S.)$$

Here $$\operatorname*{Lt}\frac{u_{n+1}}{u_n}=\operatorname*{Lt}\frac{x^{n+2}[\log(n+2)]^q}{x^{n+1}[\log(n+1)]^q}=x,$$

whatever q may be, $+ve$. or $-ve$.

The series converges for $x<1$, and diverges for $x>1$.

This test fails when $x=1$.

Now if $x=1$, Lt $n\,u_n=$Lt $n\,(\log n)^q\to\infty$,

whatever q is, $+ve.$ or $-ve.$

$\therefore$ the series diverges for $x=1$.

Hence the series converges for $x<1$, and diverges for $x\geqslant 1$ (q being positive or negative).

7·65. *If it be possible to express the ratio of the successive terms, in a positive-term series, in the form*

$$\frac{u_n}{u_{n+1}}=\alpha+\frac{\beta}{n}+\frac{\mu_n}{n^\lambda},$$

where α, β are constants, $\lambda>1$ and μ_n remains finite for all values of n, then the series Σu_n converges if $\alpha>1$, and diverges if $\alpha<1$.

Also the series $\begin{cases}\text{converges if } \beta>1\\ \text{diverges if } \beta\leqslant 1\end{cases}$ *when* $\alpha=1$.

By D'Alembert's test follows the convergence or divergence, according as $\alpha>1$ or $\alpha<1$.

If $\alpha=1$, this test fails. Then Raabe's test gives convergence or divergence, according as $\beta>$ or <1. If $\beta=1$, this test also fails.

To discuss the case when $\beta=1$, apply DeMorgan's and Bertrand's test, *i.e.*, Lt $\left\{\left(\frac{u_n}{u_{n+1}}-1\right)n-1\right\}\log n<1$ gives divergence.

In our case, this limit is equal to Lt $\left\{\mu_n\frac{\log n}{n^{\lambda-1}}\right\}$ which is zero,

for $|\mu_n|<$a constant k, for all values of n.

$\therefore$ the series diverges. Hence the theorem.

This test is due to Gauss, and it covers most of the positive-term series which arise in practice.

Examples VII (*e*)

1. Determine the convergence of the series

$$1+\frac{(1\,!)^2}{2\,!}x+\frac{(2\,!)^2}{4\,!}x^2+\frac{(3\,!)}{6\,!}x^3+\ldots$$

Leaving the first term, $u_n=\frac{(n\,!)^2}{2n\,!}x^n$,

$$\therefore\quad\frac{u_{n+1}}{u_n}=\frac{n+1}{2(2n+1)}x.$$

Consequently $\underset{n\to\infty}{\text{Lt}}\ \frac{u_{n+1}}{u_n}=\frac{x}{4}$.

Hence the series

converges if $x<4$,

and diverges if $x>4$.

Let $x=4$, then

$$\underset{n\to\infty}{\text{Lt}}\ n\left(\frac{u_n}{u_{n+1}}-1\right)=\underset{n\to\infty}{\text{Lt}}\ \frac{-n}{2n+1}=-\tfrac{1}{2}.$$

So the series is divergent if $x=4$.

Determine the behaviour of the series

2. $$\frac{3}{4}\cdot\frac{x}{5}+\frac{3.6}{4.7}\cdot\frac{x^2}{8}+\frac{3.6.9}{4.7.10}\cdot\frac{x^3}{11}+\cdots$$

3. $$\frac{1.7}{2.4}+\frac{1.3.11}{2.4.6}+\ldots+\frac{1.3\ldots(2n-1)(4n+3)}{2.4\ldots2n(2n+2)}+\ldots(St.\ Cath.)$$

4. Examine the character of the series

$$\Sigma\left\{\frac{1.3\ldots(2n-1)}{2.4\ldots2n}\right\}^a.$$

5. Test the convergence of the series whose nth term is u_n, where u_n is equal to

(a) $\frac{\sqrt{n+1}}{\sqrt{n^3-1}}x^n$,

(b) $\frac{5^2.10^2.15^2\ldots(5n)^2}{6.9.11.14.16.19\ldots(5n+1)(5n+4)}x^n$ [*Corpus*]

6. Determine the character of the series

$$\Sigma\frac{(n+1)(n+2)}{(n-1)(n-2)}.$$

7. Test the convergency and divergency of the series

$$\frac{x}{1}+\frac{1}{2}\frac{x^2}{3}+\frac{1}{2}\cdot\frac{3}{4}\frac{x^3}{5}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\frac{x^4}{7}+\ldots\ldots$$ (*P. U. 1940*)

8. Show that the series

$$1+\frac{1+a}{1+b}+\frac{(1+a)(2+a)}{(1+b)(2+b)}+\cdots$$

converges if $b\geq a+1$, and diverges if $b\leq a+1$.

9. Examine the Hypergeometric series

$$1+\frac{\alpha.\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3+\ldots$$

Solution : $\dfrac{u_{n+1}}{u_n}=\dfrac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)}x\to x$, as $n\to\infty$.

$\therefore$ the series converges if $x<1$ and diverges if $x>1$.

If $x=1$,

$$n\left(\frac{u_n}{u_{n+1}}-1\right)=\frac{n(\gamma+1-\alpha-\beta)+\gamma-\alpha\beta}{n^2+n(\alpha+\beta)+\alpha\beta}\cdot n$$
$$\to\gamma+1-\alpha-\beta, \qquad \text{as } n\to\infty.$$

So the series converges if

$$\gamma+1-\alpha-\beta>1, \qquad i.e., \text{ if } \gamma>\alpha+\beta,$$

and diverges if $\gamma<\alpha+\beta$.

If $\gamma=\alpha+\beta$,

$$\log n\left[n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right]=\frac{-(n+1)\alpha\beta.\log n}{n^2+n(\alpha+\beta)+\alpha\beta}\to 0.$$

Consequently, the series diverges. Thus the series

converges if $\gamma>\alpha+\beta$
and diverges if $\gamma\leqslant\alpha+\beta$ } when $x=1$.

Another Proof.

$$\frac{u_n}{u_{n+1}}=\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)\left(1+\frac{\alpha}{n}\right)^{-1}\left(1+\frac{\beta}{n}\right)^{-1}$$
$$\sim\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)\left[1-\frac{\alpha}{n}+0\left(\frac{1}{n^2}\right)\right]\left[1-\frac{\beta}{n}+0\left(\frac{1}{n^2}\right)\right]$$
$$=1+\frac{1}{n}\left(\gamma+1-\alpha-\beta\right)+\frac{1}{n^2}(\gamma-\alpha-\beta-\alpha\gamma-\beta\gamma+\alpha\beta)+0\left(\frac{1}{n^3}\right).$$

The result follows by Gauss's Test.

10. Prove that the series

$$1+\frac{a+1}{b+1}+\frac{(a+1)(2a+1)}{(b+1)(2b+1)}+\frac{(a+1)(2a+1)(3a+1)}{(b+1)(2b+1)(3b+1)}+\cdots$$

converges if $b>a>0$, and diverges if $a\geqslant b>0$.

11. Discuss the nature of the series

$$1+\frac{2x}{2!}+\frac{3^2x^2}{3!}+\frac{4^3x^3}{4!}+\cdots$$

$$\frac{u_{n+1}}{u_n}=\frac{(n+1)^n}{n^{n-1}}\cdot\frac{n!}{(n+1)!}x=\left(1+\frac{1}{n}\right)^n\frac{1}{1+\frac{1}{n}}\quad x\to ex, \text{ as } n\to\infty.$$

Thus the series

$$\text{converges if } x<\frac{1}{e},$$

$$\text{diverges if } x>\frac{1}{e}.$$

If $x=\frac{1}{e}$,

$$\log\frac{u_n}{u_{n+1}}=\log\frac{e}{\left(1+\frac{1}{n}\right)^{n-1}}=1-(n-1)\log\left(1+\frac{1}{n}\right)$$

$$=1-(n-1)\left[\frac{1}{n}-\tfrac{1}{2}\frac{1}{(n+\theta)^2}+\cdots\right],\ 0<\theta<1$$

$$=\frac{1}{n}+\frac{n}{2(\theta+n)^2}-\frac{1}{2(\theta+n)^2}.$$

$$\therefore \lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}}\right)=\frac{3}{2}>1.$$

So the series converges.

12. Show that the series

$$x+\frac{2^2x^2}{2\,!}+\frac{3^3x^3}{3\,!}+\frac{4^4x^4}{4\,!}+\cdots$$

converges if $x<\frac{1}{e}$, and diverges if $x\geqslant\frac{1}{e}$.

13. Prove that the series $\Sigma\frac{n\,!}{(n+1)^n}x^n$ converges if $x<e$, and diverges if $x\geqslant e$.

14. Examine the behaviour of the series

$$(a+1)\frac{x}{1!}+(a+2)^2\frac{x^2}{2\,!}+(a+3)^3\frac{x^3}{3\,!}+\cdots$$

15. Prove that the series

$$\frac{(a+x)}{1\,!}+\frac{(a+2x)^2}{2\,!}+\frac{(a+3x)^3}{3\,!}+\cdots$$

converges if $ex<1$, and diverges if $ex\geqslant 1$.

Series of terms of arbitrary signs

7·7. We now turn to series of terms of arbitrary signs. A series, in which all the terms after a certain stage are of the same sign, reduces to the preceding case. We may, therefore, consider series in which there are infinite number of positive and negative terms.

Let $$u_1+u_2+\ldots+u_n+\ldots \quad \ldots(1)$$
be such a series. Let $|u_n|=a_n$ and consider the series
$$a_1+a_2+\ldots+a_n+\ldots \quad \ldots(2)$$

Def.—If Σa_n be convergent, the series Σu_n, is said to be **absolutely convergent.**

If Σa_n is divergent, but Σu_n, is convergent, then Σu_n is said to be **conditionally convergent.**

7·71. *An absolutely convergent series is convergent.*

The statement means that if a-series converges, so does u-series. To prove the statement, we form a new series
$$(u_1+a_1)+(u_2+a_2)+\ldots+(u_n+a_n)+\ldots \quad \ldots(3)$$
so
$$u_n+a_n=0, \text{ or } 2a_n.$$

Hence if the series (2) converges, so does (3).

Let S_n, S'_n, S''_n be the sums of n terms of the series (1) (2), (3) respectively.

Obviously
$$S_n=S''_n-S'_n, \quad \ldots(4)$$

Since (2) is convergent, so is (3). If S'', S' be the limits approached by S''_n, S'_n, the above relation shows that S_n approaches a limit S, so that
$$S=S''-S'.$$

The above result also proves that *the positive as well as the negative terms of series,* (1) *separately form two convergent series.*

7·72. *In a conditionally convergent series Σu_n, the positive as well as the negative terms form two divergent series, the terms of either being taken in the order in which they occur.*

Let $v_n=u_n$ or 0, according as $u_n \gtrless 0$,

and $w_n=-u_n$ or 0, according as $u_n \lessgtr 0$.

Then evidently v_n and w_n are positive,

and $u_n=v_n-w_n$, while $|u_n|=a_n=v_n+w_n$.

Since Σu_n is conditionally convergent,

Σv_n and Σw_n both diverge to $+\infty$,

for they cannot both converge, as that would involve the convergence of $\Sigma|u_n|$; nor can one converge and the other diverge, as $\sum_1^r u_n=\sum_1^r v_n-\sum_1^r w_n$ would then diverge with n, which is contrary to the hypothesis.

Thus the sum of a conditionally convergent series is the limit of two functions of n, each of which tends to infinity with n.

7·73. (*i*) Tests of convergence for absolutely convergent series.

Since an absolutely convergent series is convergent, this case reduces to the case of a series of positive terms.

(*ii*) Tests of convergence for conditionally convergent series.

If $\phi(n)$ is a positive steadily decreasing function of n which tends steadily to zero as $n \to \infty$, then the alternating series

$$\phi(0) - \phi(1) + \phi(2) - \ldots$$

is convergent, and its sum lies between $\phi(0)$ and $\phi(0) - \phi(1)$.

Let $$S_n = \phi(0) - \phi(1) + \ldots + (-1)^n \phi(n).$$

Then $$S_{2n+1} - S_{2n-1} = \phi(2n) - \phi(2n+1) \geqslant 0,$$

and $$S_{2n} - S_{2n-2} = -\{\phi(2n-1) - \phi(2n)\} \leqslant 0.$$

Hence the sequence (S_{2n}) is decreasing and, therefore, tends to a finite limit or to $-\infty$, and the sequence (S_{2n+1}) is increasing and, therefore, tends to a finite limit or to $+\infty$.

But since

$$S_{2n} - S_{2n+1} = \phi(2n+1) > 0,$$

and Let $(S_{2n+1} - S_{2n}) = \text{Lt}\ (-1)^{2n+1}\ \phi(2n+1) = 0,$

the sequences (S_{2n}) and (S_{2n+1}) must tend to limits, and the limits must be the same. Since $S_0 = \phi(0)$ and $S_1 = \phi(0) - \phi(1)$, the limit lies between these two numbers.

Note (*i*). It is easy to see that if $\phi(n)$ *tends steadily to a positive limit,* then the series $\sum_0^n (-1)^n \phi(n)$ oscillates finitely.

Note (*ii*). There are no 'comparison tests' for conditionally convergent series.

Power-Series of a single variable

7·8. We consider a power-series of the form $\Sigma a_n x^n$.

If $\underset{n \to \infty}{\text{Lt}}\ a_n^{\frac{1}{n}} = l$, then by Cauchy's test, the power-series converges absolutely when $|x| < \frac{1}{l}$; but it cannot converge if $|x| > \frac{1}{l}$, for then Lt $|a_n x^n| > 1$, and so there will be an infinity of terms in the series whose absolute values are greater than 1.

Thus any power-series has an interval $\left(-\frac{1}{l}, \frac{1}{l}\right)$ within which it converges absolutely and outside which convergence is impossible. If $l = 0$, the series is convergent for all values of x.

Examples VII (f)

1. Show that the series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots$$

is conditionally convergent.

Show that the series whose generic terms are as follows converge conditionally :—

2. $\dfrac{(-1)^{n-1}}{\sqrt{n}}$. 3. $\dfrac{(-1)^{n-1}}{p+n}$. 4. $\dfrac{(-1)^{n-1}}{\sqrt[3]{n}+\sqrt[3]{p}}$. 5. $\dfrac{(-1)^{n-1}}{\sqrt[3]{n^2}+\sqrt[3]{p^2}}$.

6. $\dfrac{(-1)^{n-1}}{(\sqrt[3]{n}+\sqrt[3]{p})^3}$.

7. Discuss the convergence of the series

$$1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$$

8. Show that the series $\Sigma(-1)^n\ (n+a)^{-p}$, where $a>0$, is absolutely convergent, if $p>1$; conditionally convergent, if $0<p\leqslant 1$.

Investigate the behaviour of series :—

9. $\Sigma\,(-1)^n\ \sin\dfrac{a}{n}$.

The series whose generic term is $\sin\dfrac{a}{n}$ is divergent as

$\lim\limits_{n\to\infty}\left(\sin\dfrac{a}{n}\right)\Big/(a/n)=1$ and $\Sigma\ (a/n)$ is divergent. Also $\sin(a/n)$ steadily decreases to zero. Thus all conditions of the alternating series are satisfied. The series is therefore convergent.

10. $\Sigma\left(1-\cos\dfrac{a}{n}\right)$. 11. $\Sigma n\left(1-\cos\dfrac{a}{n}\right)$.

12. Show that the series whose generic term is

$$(-1)^n\frac{(\log n)^{10}}{\sqrt{n}}$$

converges.

Let $f(x)=\dfrac{(\log x)^{10}}{x^{\frac{1}{2}}}$,

Then $f'(x)=\dfrac{(\log x)^9}{x^{3/2}}\left[10-\frac{1}{2}\log x\right]$.

It follows that for value of $x > e^{20}$, $f'(x)$ is negative, and, so $f(x)$ is a steadily decreasing function of x.

Also $\underset{x\to\infty}{\text{Lt}} \dfrac{(\log x)^{10}}{\sqrt{x}} = 0$, which can be seen by following the method of evaluating the indeterminate form (∞/∞).

Thus $\dfrac{(\log x)^{10}}{\sqrt{x}}$ satisfies the conditions of alternating series.

13. Prove that the alternating series

$$\frac{2}{1}-\frac{3}{2}+\frac{4}{3}-\frac{5}{4}+\frac{6}{5}-\frac{7}{8}+\dots$$

oscillates. *(P. U. 1941)*

$$u_n=(-1)^{n-1}\frac{n+1}{n},$$

$$s_{2n}=\left(\frac{2}{1}-\frac{3}{2}\right)+\left(\frac{4}{3}-\frac{5}{4}\right)+\dots+\left(\frac{2n}{2n-1}-\frac{2n+1}{2n}\right)$$

$$=\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+\dots+\frac{1}{2n(2n-1)}.$$

Comparing this series with $\Sigma\dfrac{1}{n^2}$ which is convergent, it follows that the sequence s_{2n} converges to a limit s.

But $\quad s_{2n+1}=s_{2n}+\dfrac{2n+2}{2n+1}$;

$\therefore \underset{n\to\infty}{\text{Lt}}\ s_{2n+1}=s+1$

Thus the series oscillates between s and $s+1$.

14. Prove that the series $\sum_1^\infty(-1)^{n+1}\,a_n$ diverges if

$$a_n=\frac{1}{\sqrt{n}}+\frac{(-1)^{n-1}}{n},$$

although the terms are alternately positive and negative, and approach zero.

$$u_n=\frac{(-1)^{n+1}}{\sqrt{n}}+\frac{1}{n},$$

$$s_n=\left[\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}\dots+\frac{(-1)^{n+1}}{\sqrt{n}}\right]+\left[1+\frac{1}{2}+\dots+\frac{1}{n}\right].$$

The series within the first bracket converges, being an alternating series, and the series within the second bracket diverges.

It will be seen that *all* the conditions of the alternating series are not satisfied, *viz.*, "u_n *decreases to zero, steadily*," is not satisfied.

15. Show that the series $\Sigma\frac{(-1)^n}{\sqrt{n}+(-1)^n}$ diverges, although the terms are positive and negative and the nth term tends to zero.

$$\left[\text{Here } u_n=\frac{(-1)^n}{\sqrt{n}+(-1)^n}=\frac{(-1)^n}{\sqrt{n}}+\frac{(-1)^n}{\sqrt{n}}\right.$$
$$\left.=\frac{(-1)^n}{\sqrt{n}}-\frac{1}{n+(-1)^n\sqrt{n}}\right].$$

*Arithmetic Operations on Series

7·9. It has already been emphasized that the term 'sum', used to designate the limit of the sequence of partial sums of a series, is misleading, in so far as it arouses a belief that an infinite series may be operated on by the same rules as an actual sum of a definite number of terms, *i.e.*, of the form $a+b+c+d$. This is not the case, however ; although some of the rules in question do actually remain valid for infinite series. The principal laws are the associative, distributive and commutative laws.

The following theorems are intended to show how far these laws remain true for infinite series.

7·911. *The associative law holds unrestrictedly for convergent infinite series.*

Let $\quad A=a_1+a_2+\ldots$ be convergent,

and let $\quad b_1=a_1+a_2+\ldots+a_{m_1}$, $b_2=a_{m_1+1}+\ldots+a_{m_2},\ldots$

Then the series $B=b_1+b_2+b_3+\ldots$ is convergent, and $A=B$.

For $B_n=A_{m_n}$ and $\underset{n\to\infty}{\text{Lt}}\ A_{m_n}=A$.

It may be noted that the number of terms which b_n embraces may increase indefinitely with n. We have proved that

$$a_1+a_2+a_3+\ldots=A$$

implies $(a_1+a_2+\ldots+a_{m_1})+(a_{m_1+1}+\ldots a_{m_2})+\ldots=A$.

This theorem relates to *the insertion of brackets.*

7·912. $B=b_1+b_2+b_3+\ldots$ *is convergent,*

and $\quad b_1=a_1+a_2+\ldots+a_{m_1}$, $b_2=a_{m_1+1}+\ldots+a_{m_2},\ldots$

(*i*) *If* $A=a_1+a_2+\ldots$ *is convergent, then* $A=B$.

(*ii*) *If the terms* $a_n>0$, A *is convergent.*

(*iii*) *If each* $m_n-m_{n-1}\leqslant p$, *a constant, and* $a\to 0$;
then A *is convergent.*

To prove (*i*), we apply 7·911, and show that $A=B$.

To prove (*ii*), we notice that

$$\sum_{m+1}^{\infty} b_n < \epsilon \text{ for } n \geq m.$$

$$\therefore \quad \sum_{1}^{\infty} b_n - \sum_{1}^{s} a_n < \epsilon \text{ for } s \geq m_n.$$

To prove (*iii*), we may set

$$A_n = B_r + b'_{r+1},$$

where b'_{r+1} denotes a part of the a terms in b_{r+1}.

Since b_{r+1} contains at most p terms of A, $b'_{r+1} \to 0$.

Hence Lt A_s = Lt B_r, or $A = B$.

This theorem relates to the removal of brackets.

It may be noted that we cannot without consideration omit brackets occurring in a series, as the following example shows :—

The series $0+0+0+\ldots$ is certainly convergent and has the sum 0. If we substitute everywhere $(1-1)$ for 0, we obtain the correct equality

$$(1-1)+(1-1)+\ldots \equiv \Sigma(1-1)=0.$$

But by omitting the brackets we obtain the oscillating series

$$1-1+1-1+\ldots$$

Grouping the terms in a slightly different way, we obtain

$$1-(1-1)-(1-1)\ldots \equiv 1-0-0\ldots \to 1.$$

$$\therefore \quad 0=1.$$

In former times, Mathematicians found themselves fairly at a loss when confronted with paradoxes like this.

It is said that Guido Grandi believed that in the above erroneous train of argument which turns 0 into 1, he had obtained "a Mathematical proof of the possibility of the creation of the world from nothing ! "

7·913. *Convergent series may be added, or subtracted term by term.*

$$A = a_1 + a_2 + a_3 + \ldots, \text{ and } B = b_1 + b_2 + b_3 + \ldots$$

imply $C = \sum_{n=1}^{\infty} (a_n \pm b_n) = A \pm B$, and also, without brackets,

$$a_1 + b_1 + a_2 + b_2 + \ldots = A + B.$$

For, $C_n = A_n \pm B_n$. Now pass to the limit.

7·914. *Convergent series may be multiplied by a constant.*

If $A = a_1 + a_2 + a_3 + \ldots$, and λ is an arbitrary constant, then $\Sigma \lambda a_n = \lambda A$. It is already proved.

These theorems not only provide the convergence of the new series, but also set up a relation between its sum and that of the old series.

Ex. The series $\sum_1^\infty \frac{(-1)^{n-1}}{n}$ is convergent. Let s be its sum.

By 7·911, the series $\sum_{k=1}^\infty \left(\frac{1}{2k-1}-\frac{1}{2k}\right)$

and $\sum_{k=1}^\infty \left(\frac{1}{4k-3}-\frac{1}{4k-2}+\frac{1}{4k-1}-\frac{1}{4k}\right)$

are convergent and also each has its sum s.

By 7·914, multiply the 1st by $\frac{1}{2}$;

$$\therefore \quad \sum_{k=1}^\infty \left(\frac{1}{4k-2}-\frac{1}{4k}\right)=\frac{s}{2}.$$

Add this series, term by term, to the 2nd, and we obtain

$$\sum_{k=1}^\infty \left\{\frac{1}{4k-3}+\frac{1}{4k-1}-\frac{1}{2k}\right\}=\frac{3}{2}s,$$

i.e., $$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\cdots=\frac{3}{2}\log 2.$$

This shows that if the series $\log 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ be deranged, the sum of the resulting series is affected, *i.e.*, the *commutative law is not likely to hold always.*

We proceed to examine this now.

7·92. Does a derangement of a convergent series affect its sum?

A convergent series may consist of positive terms only, or it may be absolutely convergent, or conditionally convergent. So we heve to discuss three cases :

(*a*) A positive-term convergent series.

(*i*) Let $u_1+u_2+u_3+\ldots u_n+\ldots$ be the positive-term series whose sum is s, and

(*ii*) let $v_1+v_2+v_3+\ldots+v_n+\ldots$ be a series whose terms are the same as those of the 1st, arranged in any new order, *i.e.*, every v is a u and every u, a v.

Let s'_m be the sum of the 1st m terms of the series (*ii*).

Since all these terms occur in the series (*i*), it is evident, n may be chosen so large that the 1st m terms of the series (*ii*) are to be found among the 1st n terms of the series (*i*).

Hence $s'_m < s_n < s$,

which shows that the sesies (*ii*) converges, and its sum $s' \leqslant s$.

In a similar manner, $s_k < s'_r$,

i.e., $s \leqslant s'$. Hence $s'=s$.

(*b*) **An absolutely convergent series.**

Let the series (*i*), *i.e.*, Σu_n be absolutely convergent, converging to s, and the series (*ii*), *i.e.*, Σv_n be the deranged series.

Then $\Sigma \mid u_n \mid$ is convergent ; let its sum be σ.

Now the series $\Sigma \{ u_n + \mid u_n \mid \}$ contains no negative term, and no term in it is greater than twice the corresponding term in the series $\Sigma \mid u_n \mid$.

Since $\Sigma \mid u_n \mid$ is convergent,

$\Sigma \{u_n + \mid u_n \mid \}$ converges to S, not greater than 2σ.

Also $$S = s + \sigma, \text{ (Art. 7·193).}$$

Suppose now $\Sigma v_n = s'$, $\Sigma \mid v_n \mid = \sigma'$, and $\Sigma \{v_n + \mid v_n \mid \} = S'$;

then $$S' = s' + \sigma,$$

But by (*a*), $\sigma = \sigma'$ and $S = S'$,

$$\therefore \quad s' = S' - \sigma' = S - \sigma = s.$$

Hence the sum is not affected by derangement.

Otherwise : Let Σu_n be absolutely convergent and Σv_n be the series formed by derangement.

Take m so large that

$$R = \mid a_{m+1} \mid + \mid a_{m+2} \mid + \ldots < \epsilon.$$

We may now take n so large that $\mu \equiv (\overset{n}{\underset{1}{\Sigma}} u_n - \overset{n}{\underset{1}{\Sigma}} v_n)$ contains no term whose index is $\leqslant m$. Thus the terms of μ taken with positive signs are a part of R, and hence $\mu < R < \epsilon$, for $n > m$.

$\therefore$ Σv_n is convergent and $\Sigma v_n = \Sigma u_n$. The same reasoning shows that $\Sigma \mid v_n \mid$ is convergent. Hence Σv_n is absolutely convergent and the sum remains unaltered.

(*c*) *A conditionally convergent series can be made to converge by a suitable derangement to any pre-assigned number a, whatever.*

Suppose Σu_n is conditionally convergent. Since the sums of positive and negative terms tend to $+\infty$ and $-\infty$ respectively, we form a new series, whose sum is a, in the following manner :—

Take positive terms from the given series in the order in which they occur in it, until their sum exceeds a. Add to these, in the order in which they occur in the given series, negative terms, until the total sum $< a$. Again, beginning with the positive terms where we left off, add positive terms until the total sum $> a$. Then return to the negative terms, and so on. It is clear that the sum of the 1st n terms of the new series thus obtained is, alternately, greater than and less than a, and that it differs from a by a number which approaches zero as its limit, as Σu_n is convergent (*Goursat*).

Hence we have proved that a derangement does not alter the sum of (*i*), a positive-term convergent series, and [*ii*), an absolutely convergent series ; but it may affect the sum of a conditionally convergent series.

Note. We know that if $f(x)$ be a positive function, decreasing to zero as x increases, then the series $\sum_{1}^{\infty}(-)^{n-1}f(n)$ is convergent. Let s be its sum, and let this series be deranged.

If $\underset{n\to\infty}{Lt}\ nf(n)=g$, *and if* k *is the limit of the ratio of the number of positive to the number of negative terms, the sum of the deranged series is* $s+\frac{1}{2}g\log k$.

The proof of this theorem is beyond the scope of the book.

Multiplication of Infinite Series

7·93. We now consider the multiplication of two infinite series, and enquire to what extent the distribution law holds for them. We have already seen that a convergent series may be multiplied term by term by a constant, without affecting its convergence. The student can easily deduce from the distributive law : $a(b+c)=ab+ac$, that

$$(a_1+a_2+a_3+\ldots+a_m)(b_1+b_2+\ldots+b_n)=a_1b_1+a_1b_2+\ldots+a_nb_n,$$

or that $\left(\sum_{\lambda=1}^{m} a_\lambda\right)\left(\sum_{\mu=1}^{n} b_\mu\right)=\Sigma a_\lambda b_\mu$, $\quad\left[\begin{matrix}\lambda=1,\ 2,\ 3,\ldots\ m,\\ \mu=1,\ 2,\ 3,\ldots\ n,\end{matrix}\right.$

where on the right, λ and μ take up, independently of one another, all the integral values from 1 to m, and 1 to n respectively and all such products, $a_\lambda b_\mu$, mn in number, are to be added, in any order, whatsoever.

'Does this result continue to hold, when $m\to\infty$, $n\to\infty$, *i.e.*, can the distributive law be extended, without change, to infinite series ?' is the problem to be considered.

It may be noted that $\Sigma a_\lambda b_\mu$, when λ and μ run through all the integral values from 1 to ∞, independently of one another, denotes the infinite series whose terms are all possible products $a_\lambda . b_\mu$, without repetition. If we take two rectangular axes, the points whose co-ordinates are positive integers or zero (*i.e.*, $x=\lambda$, $y=\mu$), are called lattice points of a non-negative lattice system. Thus to each term $a_\lambda b_\mu$ of the infinite series $\Sigma a_\lambda b_\mu$, corresponds a lattice point, and conversely. Further,

we may write the terms of the series $\Sigma a_\lambda b_\mu$ in the form (+signs are omitted) :—

$$\begin{array}{lllll} a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 & a_1b_5\ldots \\ a_2b_1 & a_2b_2 & a_2b_3 & a_2b_4 & a_2b_5\ldots \\ a_3b_1 & a_3b_2 & a_3b_3 & a_3b_4 & a_3b_5\ldots \\ a_4b_1 & a_4b_2 & a_4b_3 & a_4b_4 & a_4b_5\ldots \\ a_5b_1 & a_5b_2 & a_5b_4 & a_5b_4 & a_5b_5\ldots \\ \ldots\ldots\ldots\

7·932. By our theorem of the preceding article, we have established that the distributive law may, at any rate, be extended without change to infinite series—with an arbitrary arrangement of the products $a_\lambda b_\mu$—if the given series are absolutely convergent. To prove the convergence of the product-series, under less stringent conditions than that of absolute convergence of the series Σa_n and Σb_n, is beyond the scope of this book, and we shall not, therefore, enter into any further investigation of the question of multiplication of series, but we shall show by means of an example due to Cauchy, that the distributive law does not, in general, holds, so that the product of two such series Σa_n and Σb_n may no longer be formed according to the elementary rules.

Ex. Consider the series

$$s=\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots,$$

$$\sigma=\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots\equiv s.$$

The series s, being alternating, is conditionally convergent.

Arranging the terms of the product as in (4), Art. 7·931, we have

$$S=\frac{1}{\sqrt{1}\ \sqrt{1}}-\left\{\frac{1}{\sqrt{1}\ \sqrt{2}}+\frac{1}{\sqrt{2}\ \sqrt{1}}\right\}+\left\{\frac{1}{\sqrt{1}\ \sqrt{3}}+\frac{1}{\sqrt{2}\ \sqrt{2}}+\frac{1}{\sqrt{3}\ \sqrt{1}}\right\}-\ldots$$

Let t_n denote its nth term. Then

$$t_n=(-)^{n-1}\left\{\frac{1}{\sqrt{1}\ \sqrt{n}}+\frac{1}{\sqrt{2}\sqrt{n-1}}+\ldots+\frac{1}{\sqrt{n}\ \sqrt{1}}\right\}.$$

Replacing each root in the denominators by the largest $\sqrt{n}$, it follows that $|t_n|\geqslant\frac{n}{\sqrt{n}\ \sqrt{n}}=1$, and, therefore, the product-series S does not converge. This shows that *our theorem does not hold when both the series are conditionally convergent.*

Examples VII (g)

1. Show that $1+\frac{1}{3}-\frac{1}{2}-\frac{1}{4}+\frac{1}{5}+\frac{1}{7}-\frac{1}{6}-\frac{1}{8}+\ldots=\log 2$.
2. Show that $1+\frac{1}{3}+\frac{1}{5}-\frac{1}{2}-\frac{1}{4}+\ldots=\frac{1}{2}\log 6$.

   Given $\sum_1^\infty\frac{1}{n^2}=\frac{\pi^2}{6}$, show that

3. $1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots=\frac{\pi^2}{8}$.

4. $1+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{13^2}+\ldots=\frac{\pi^2}{9}$.

5. $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots=\frac{\pi^2}{12}$.

6. By the multiplication of series, prove the following relations :—

(*i*) $(1+x+x^2+x^3+\ldots)^2=(1+2x+3x^2+4x^3+\ldots)=(1-x)^{-2}$,

(*ii*) $e^x.e^y=e^{x+y}$, (*iii*) $\cos^2 x+\sin^2 x=1$,

(*iv*) $\sin 2x=2\sin x\cos x$,

(*v*) $\frac{1}{2}(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots)^2=\frac{1}{2}-\frac{1}{3}(1+\frac{1}{2})+\frac{1}{4}(1+\frac{1}{2}+\frac{1}{3})-\frac{1}{5}(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4})+\ldots$

7. Show that $\sin(x+y)=\sin x\cos y+\cos x\sin y$, given that

$$\sin x=\sum_1^\infty (-)^{n-1}\frac{x^{2n-1}}{(2n-1)!}, \text{ and } \cos x=\sum_1^\infty (-)^n\frac{x^{2n}}{(2n)!}.$$

Examples VII (*h*)

Examine the convergence or divergence of the following series :—

1. $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\ldots$

2. $\frac{1^2}{2^2}+\frac{1^2.3^2}{2^2.4^2}x+\frac{1^2.3^2.5^2}{2^2.4^2.6^2}x^2+\ldots$

3. $x^2+\frac{2^2}{3.4}x^4+\frac{2^2.4^2}{3.4.5.6.}x^6+\frac{2^2.4^2.6^2}{3.4.5.6.7.8}x^8+\ldots$

4. $\frac{1}{a}+\frac{bx}{a^2}+\frac{b^2x^2}{a^3}+\ldots$

5. Show that the series

$$\frac{\sin x}{1^2}-\frac{\sin 2x}{2^2}+\frac{\sin 3x}{3^2}-\frac{\sin 4x}{4^2}+\ldots$$

converges absolutely for any value of x.

6. Show that the following series converges absolutely for any value of $|x|<1$, no matter what θ may be,

$$1+x\sin\theta+x^2\sin 2\theta+x^3\sin 3\theta+\ldots$$

7. Investigate the behaviour of the series $\Sigma\frac{1}{n^2-x}\left(x>0\right)$.

8. Examine the convergence of the series

$$\frac{x}{1+x}+\frac{3x^3}{1+3x}+\frac{5x^5}{1+5x}+\frac{7x^7}{1+7x}+\ldots$$

for all values of x.

9. Show that the series $1+\frac{x}{x^2-1}+\frac{x^2}{x^4-1}+\frac{x^4}{x^8-1}+\dots$ converges for all values of x, except $x=\pm 1$.

10. Prove that the series $\frac{2}{1^2}-\frac{3}{2^2}+\frac{4}{3^2}-\frac{5}{4^2}+\dots$ is conditionally convergent.

11. Prove that the series

$$(\log 3-\log 2)-(\log 4-\log 3)+(\log 5-\log 4)-\dots$$

is convergent, but the series

$$\log 3-\log 2-\log 4+\log 3+\log 5-\log 4-\dots$$

is not convergent.

12. If $\frac{u_n}{u_{n+1}}=\frac{n^k+An^{k-1}+Bn^{k-2}+Cn^{k-3}+\dots}{n^k+an^{k-1}+bn^{k-2}+cn^{k-3}+\dots}$, when k is a+ve. integer, show that the series Σu_n is convergent if $A-a-1>0$, and divergent if $A-a-1\leqslant 0$.

13. Prove that the sum to infinity of the series

$$1-\tfrac{1}{2}-\tfrac{1}{4}+\tfrac{1}{3}-\tfrac{1}{6}-\tfrac{1}{8}+\tfrac{1}{5}-\dots \text{is } \tfrac{1}{2}\log 2.$$

Examine whether the following series are convergent or divergent :—

14. $\frac{3}{2}x+\frac{5}{5}x^2+\frac{7}{10}x^3+\frac{9}{17}x^4+\dots$

15. $\frac{x}{1+x^3}+\frac{x^2}{1+x^4}+\frac{x^3}{1+x^5}+\dots$

16. Determine the character of the series

$$\frac{x}{\sqrt{1+a}}+\frac{x^2}{\sqrt{2+a}}+\frac{x^3}{\sqrt{3+a}}+\dots$$

for all real values of x and a.

17. Show that the series $\sum_1^\infty \frac{r^n}{n}$ converges for $-1\leqslant r<1$, diverges for $1\leqslant r$, and oscillates for $r<-1$.

18. Prove that the series

$$1+nx+\frac{n(n+3)}{2\,!}x^2+\frac{n(n+4)(n+5)}{3\,!}x^3+\dots$$
$$+\frac{n(n+r+1)(n+r+2)\dots(n+2r-1)}{r\,!}x^r+\dots$$

is convergent if $x^2<(\frac{1}{4})^2$.

19. Show that the series $\Sigma\frac{\cos n\theta}{n^r}$ and $\Sigma\frac{\sin n\theta}{n^r}$ are, in general, convergent if $r>0$.

20. Prove that if Σnu_n is convergent, so also is Σu_n.

21. Show that the series $\Sigma(-1)^{n-1}u_n$ diverges if

$$u_n=\frac{1}{\sqrt{n}}+\frac{(1-)^{n-1}}{n}, \text{ or if } u_n=\frac{1}{\sqrt{n}+(-1)^{n-1}},$$

although the terms are alternately positive and negative, and tend to zero as a limit.

22. Prove that if $p\to 0$ through positive values,

$$(i)\ \operatorname*{Lt}_{p\to 0} p\sum_1^\infty \frac{1}{n^{1+p}}=1\,; \quad (ii)\ \operatorname*{Lt}_{p\to 0}\left\{\Sigma\frac{1}{n^{1+p}}-\frac{1}{p}\right\}=C,$$

where C is Euler's constant.

23. If $\sum_1^\infty u_n$ be a divergent positive-term series, prove that $\Sigma\frac{u_n}{s_n}$ is also divergent, where $s_n=\sum_1^n u_n$.

24. If Σu_n diverges, so also does $\Sigma\frac{u_n}{1+u_n}$.

25. If Σu_n is a positive-term convergent series, then $\Sigma\sqrt{u_nu_{n+1}}$ is also convergent.

26. *If ϕ_n is a positive function of n which tends steadily to zero as $n\to\infty$, and Σa_n is a series which converges or oscillates finitely, prove that the series $\sum_0^\infty a_n\phi_n$ is convergent.* (*Dirichlet*)

Let $$s_n=a_0+a_1+\ldots+a_n.$$

Then $a_0\phi_0+a_1\phi_1+\ldots+a_n\phi_n$

$\equiv s_0\phi_0+(s_1-s_0)\phi_1+\ldots+(s_n-s_{n-1})\phi_n$

$\equiv s_0(\phi_0-\phi_1)+s_1(\phi_1-\phi_2)+\ldots+s_{n-1}(\phi_{n-1}-\phi_n)+s_n\phi_n.$

Now the series $(\phi_0-\phi_1)+(\phi_1-\phi_2)+\ldots$ is a positive-term series and is convergent, for the sum to n terms is $\phi_0-\phi_n$ and Lt $\phi_n=0$.

Hence the series $\sum_0^\infty s_r(\phi_r-\phi_{r+1})$ is convergent, for we can find a constant k, so that $|s_r|<k$ for all values of r;

or simply $s_r(\phi_r-\phi_1)\leqslant\Sigma|s_r|(\phi_r-\phi_{r+1})<k\Sigma(\phi_r-\phi_{r+1})$.

$\therefore$ It is absolutely convergent.

Also $s_n\phi_n\to 0$ as $\phi_n\to 0$,

$\therefore$ the series $\sum_0^\infty a_n\phi_n$ is convergent.

27. If ϕ_n is a positive and decreasing or increasing function of n tending to a limit and Σa_n is convergent, then $\Sigma a_n\phi_n$ is convergent,

For, Lt $\phi_n=l$ (say), $\therefore$ Lt $(\phi_n-l)=0$, or Lt $(l-\phi_n)=0$.

Hence taking $\psi_n=\phi_n-l$ and applying Dirichlet's test, we get the result.

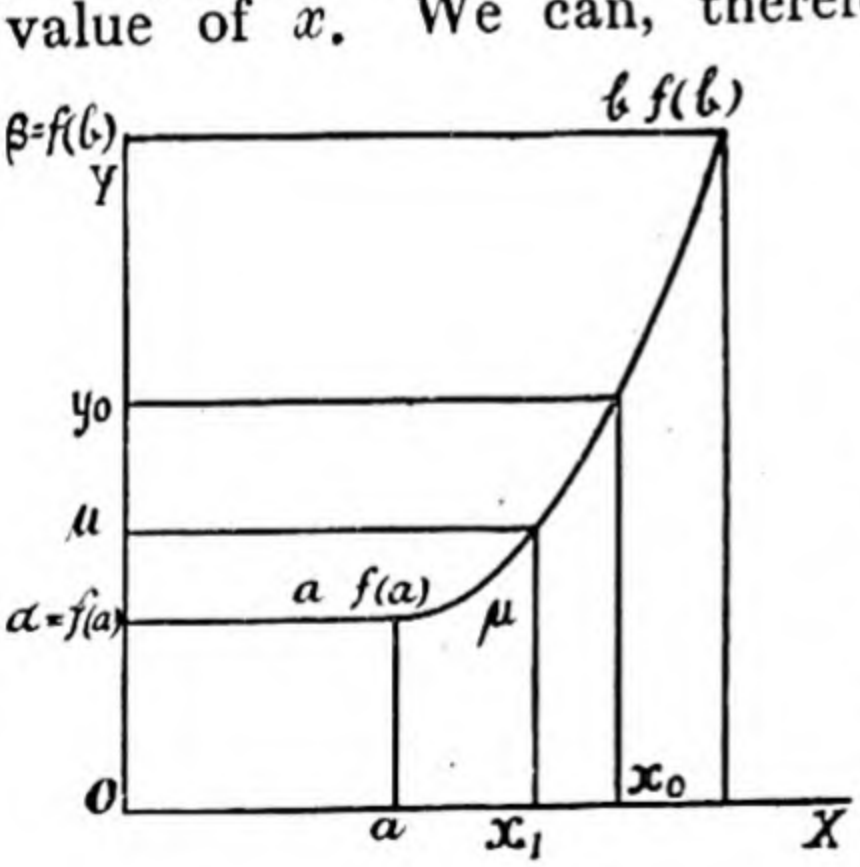
CHAPTER VIII

INVERSE CIRCULAR FUNCTIONS

8·1. Suppose $y=f(x)$ is a continuous monotonic (*i.e.*, steadily increasing or decreasing) function of x in the interval $a \leqslant x \leqslant b$.

In virtue of the continuity of the function $f(x)$, it will assume at least once every value μ between $\alpha=f(a)$ and $\beta=f(b)$.* Since the function is monotonic, the value will be assumed only once. For, if possible, let the value be assumed for two distinct values x_1 and x_2, $x_2>x_1$, of x; then $f(x_2) \gtrless f(x_1)$, according as the function is monotonically increasing or decreasing, and consequently, $f(x_1)$ and $f(x_2)$ cannot be equal. Hence y describes the closed interval between the values $\alpha=f(a)$ and $\beta=f(b)$; to each value of y there will correspond exactly one value of x. We can, therefore, think of x as a single valued function of y in this interval, and denote the relation by

$$x=f^{-1}(y).$$



We see that this function is unique.

The inverse function so obtained is also continuous. Take for definiteness, that y increases with x, and let y_0, be a value of y in the interior of the interval $\alpha<y<\beta$, then there exists a single value x_0, such that

$$y_0=f(x_0),\ a<x_0<b.$$

Let $\epsilon>0$, be an arbitrarily small number, and

$$y_0-h=f(x_0-\epsilon),\ y_0+k=f(x_0+\epsilon).$$

If δ be the smaller of the numbers h and k, then as y varies in the interval $(y_0-\delta,\ y_0+\delta)$, x will vary within the interval

*An *arithmetical proof* of this may be found is Hardy's *Pure Mathematics*.

$(x_0-\epsilon, x_0+\epsilon)$. If y_1 and y_2 are two values of y within this interval, and x_1, x_2 the corresponding values of x,

$$| f^{-1}(y_2)-f^{-1}(y_1) | = | x_2-x_1 | < 2\epsilon,$$

which proves the continuity of the function

$$x=f^{-1}(y).$$

8·11. The graph of the inverse function $y=f^{-1}(x)$

The graph of the inverse function

$$y=f^{-1}(x), \text{ i.e., } x=f(y)$$

may be obtained from that of $y=f(x)$.

If (x', y') be a point on the curve $y=f(x)$, it is seen that (y', x') satisfies the equation $x=f(y)$. But the points (x', y') and (y', x') are reflections of each other in the line $y=x$.

Thus the graph of $y=f^{-1}(x)$ can be obtained from the graph of $y=f(x)$ by taking its image in the line $y=x$.

8·2. Definitions of Inverse circular functions and their Principal values.

8·21. The function $\sin^{-1}x$. The function $\sin x$ is a continuous steadily increasing function of x in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and its values lie in the interval $(-1, 1)$. It follows therefore that the equation

$$\sin y=x \qquad -1\leqslant x\leqslant 1 \quad (i)$$

has only one root, which lies in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. This equation, therefore, defines y as a continuous, steadily increasing function of x in the interval $-1\leqslant x\leqslant 1$, and the values of y shall lie in the interval given by $-\frac{\pi}{2}\leqslant y\leqslant\frac{\pi}{2}$.

This function y we shall denote by

$$y=\text{Sin}^{-1}x, \text{ or } y=\text{ arc Sin } x.$$

The complete solution of the equation (i) when y is not restricted to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, is

$$y=n\pi+(-1)^n \sin^{-1}x,$$

where n is any integer, positive or negative. Out of these values, the value of y that lies in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, as considered above, is called the *principal value, and is numerically the smallest value of y which has the same sign as x.*

The general value of y is also sometimes denoted by $\sin^{-1}x$.

8·22. The function $\cos^{-1}x$.

As in the previous case, when $-1 \leqslant x \leqslant 1$, the equation

$$\cos y = x$$

has only one root, in the interval $(0, \pi)$, as $\cos y$ is a continuous steadily decreasing function of y in the interval $(0, \pi)$. This equation, therefore, defines y as a continuous function of x, which steadily decreases when $-1 \leqslant x \leqslant 1$; the values of y lie between $(0, \pi)$, the lower value 0, and the upper value π being assumed at $x=1$ and $x=-1$, respectively. We represent this function as

$$y = \cos^{-1}x, \text{ or } y = \text{arc cos } x,$$

the values of y belonging to the interval $(0, \pi)$ being called the *principal values.*

It will be seen, that the general solution of the above equation is

$$y = 2n\pi \pm \text{arc cos } x.$$

n being any integer. *Among the values of this set that which is numerically the smallest positive value of y, and satisfies the equations cos y=x, is called the Principal value.*

8·221. *When $\cos^{-1}x$ and $\sin^{-1}x$ have the principal values,*

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}. \quad \ldots \quad (1)$$

Proof. Calling the left-hand side y, we have

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0.$$

$\therefore$ y is constant, *i.e.*, $y=c$. To find this constant c, we give a particular value to x, say zero ; then $c = \frac{\pi}{2}$, and hence the relation.

Ex. $\sin^{-1}(-\frac{1}{2}) = -\frac{\pi}{6}$, and $\cos^{-1}(-\frac{1}{2}) = \frac{2\pi}{3}$;

thus $$\sin^{-1}(-\tfrac{1}{2}) + \cos^{-1}(-\tfrac{1}{2}) = \frac{\pi}{2}.$$

8·23. The function $\tan^{-1}x$.

The general solution of the equation

$$\tan y = x$$

is given by

$$y = n\pi + y_0,$$

where y_0 is the least value of y which satisfies the equation and has the same sign as x. This is called the *principal value of y,* and is included in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Unless otherwise stated, the value y_0 will be denoted by $\tan^{-1}x$, though the general

value is also denoted by this symbol. Thus the principal value of $\tan^{-1}x$ lies in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

8·24. The functions $\cot^{-1}x$, $\sec^{-1}x$, $\operatorname{cosec}^{-1}x$.

These functions are respectively the solutions of

$$\cot y=x,\ \sec y=x,\ \operatorname{cosec} y=x.$$

For $\sec^{-1}x$, the *principal value* of y is restricted to the interval $(0, \pi)$, and for $\cot^{-1}x$, $\operatorname{cosec}^{-1}x$, the *principal values* of y are restricted to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Some writers call the principal value of $\cot^{-1}x$ as one that lies within $(0, \pi)$.

On this understanding, we can prove that

$$\tan^{-1}x+\cot^{-1}x=\pi/2. \quad \ldots \quad (2)$$

It is also obvious that $\sec^{-1}x+\operatorname{cosec}^{-1}x=\pi/2$, $\ldots$ (3)
when x does not lie between -1 and 1.

Ex. 1. $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)=-\frac{\pi}{6}$; and $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)=\frac{2\pi}{3}$, when it is restricted to the interval $(0, \pi)$. Thus

$$\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)+\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)=\frac{\pi}{2}.$$

Ex. 2. Verify the formula (3), when (i) $x=-\sqrt{2}$, (ii) $x=-2$.

8·3. Addition and Subtraction formulae

Let $\sin^{-1}x=\theta$, $\sin^{-1}y=\phi$,

so that $\sin\theta=x$ and $\sin\phi=y$.

If θ and ϕ are both acute,

$$\cos\theta=\sqrt{1-x^2},\ \cos\phi=\sqrt{1-y^2}.$$

Since $\sin(\theta+\phi)=\sin\theta\cos\phi+\cos\theta\sin\phi$

$$\theta+\phi=\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\},$$

provided θ and ϕ, $\theta+\phi$ are all acute.

Thus $\sin^{-1}x+\sin^{-1}y=\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\}$. $\ldots$ (4)

If $\theta+\phi$ is not acute, but θ and ϕ are acute, separately,

$$\sin^{-1}x+\sin^{-1}y=\pi-\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\}.$$

To avoid the difficulty of writing double formulæ, we do not restrict the right-hand side member to its principal value, and write in general

$$\sin^{-1}x+\sin^{-1}y=\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\},$$

where the inverse function on the right is not restricted to the principal value.

We deduce similarly that

$$\sin^{-1} x-\sin^{-1} y=\sin^{-1}\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\}, \quad \ldots \quad (5)$$

$$\cos^{-1} x+\cos^{-1} y=\cos^{-1}\{xy-\sqrt{(1-x^2)(1-y^2)}\}, \quad \ldots \quad (6)$$

$$\cos^{-1} x-\cos^{-1} y=\cos^{-1}\{xy+\sqrt{(1-x^2)(1-y^2)}\}, \quad \ldots \quad (7)$$

where the inverse functions on the right are not restricted to the principal values.

8·31. *To prove that*

$$tan^{-1} x+tan^{-1} y=tan^{-1}\frac{x+y}{1-xy}, \quad \ldots \quad \ldots \quad (8)$$

$$tan^{-1} x-tan^{-1} y=tan^{-1}\frac{x-y}{1+xy}, \quad \ldots \quad \ldots \quad (9)$$

where the functions on the right are not restricted to their principal values.

Let $\tan\ \theta=x$, $\tan\ \phi=y$,

Then $$\tan(\theta+\phi)=\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}=\frac{x+y}{1-xy},$$

$$\tan^{-1} x+\tan^{-1} y=\theta+\phi=\tan^{-1}\frac{x+y}{1-xy}.$$

In this case, if $0<x<1$, $0<y<1$, the function on the right will have its principal value. If x, y are both positive, $xy>1$, the function being restricted to the principal values,

$$\tan^{-1} x+\tan^{-1} y=\pi+\tan^{-1}\frac{x+y}{1-xy}.$$

The second formula follows similarly.

Ex. Verify each of the formulæ (4) to (9) by giving suitable values to x and y.

8·32. We leave it as an exercise to the student to prove the following results

$$3\sin^{-1} x=\sin^{-1}(3x-4x^3),$$

$$3\cos^{-1} x=\cos^{-1}(4x^3-3x),$$

$$3\tan^{-1} x=\tan^{-1}\frac{3x-x^3}{1-3x^2},$$

$$\tan^{-1} x+\tan^{-1} y+\tan^{-1} z=\tan^{-1}\frac{x+y+z-xyz}{1-yz-zx-xy}.$$

8·41. *Show that* $\sin^{-1}\frac{a-c}{a+c}-2\sin^{-1}\sqrt{\frac{a}{a+c}}$ *is an odd multiple of* $\frac{\pi}{2}$. *(C. U. 1934)*

We may suppose $a>c$.

Let $\sin^{-1}\sqrt{\frac{a}{a+c}}=x$, so that $\sin x=\sqrt{\frac{a}{a+c}}>\frac{1}{\sqrt{2}}$.

$$\therefore\quad x>\frac{\pi}{4},\quad \text{and}\quad \cos x=\sqrt{\frac{c}{a+c}}.$$

$$\therefore\quad \sin 2x=2\sin x\cos x=\frac{2\sqrt{ac}}{a+c},$$

$$\text{and}\quad \cos 2x=-\left[1-\frac{4ac}{(a+c)^2}\right]^{\frac{1}{2}}=-\frac{a-c}{a+c}\left(\because\ 2x>\frac{\pi}{2}\right).$$

$$\text{Hence}\quad \sin^{-1}\frac{a-c}{a+c}-2\sin^{-1}\sqrt{\frac{a}{a+c}}=\sin^{-1}\frac{a-c}{a+c}-\sin^{-1}\frac{2\sqrt{ac}}{a+c}$$

$$=\sin^{-1}\left\{-\frac{(a-c)^2}{(a+c)^2}-\frac{2\sqrt{ac}}{a+c}\sqrt{1-\left(\frac{a-c}{a+c}\right)^2}\right\}$$

$$=\sin^{-1}\left[-\left\{\frac{(a-c)^2}{(a+c)^2}+\frac{4ac}{(a+c)^2}\right\}\right]=\sin^{-1}(-1)$$

$$=n\pi-(-1)^n\frac{\pi}{2}$$

$$=[2n-(-1)^n]\frac{\pi}{2}=(2n\pm1)\frac{\pi}{2}.$$

If $c>a$, we can write the expression in the question as

$$-\left\{\sin^{-1}\frac{c-a}{a+c}+2\sin^{-1}\sqrt{\frac{a}{a+c}}\right\}$$

and in this case, $\cos 2x=\frac{c-a}{a+c}$, $\because\ x<\frac{\pi}{4}$.

8·42. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$, prove that

$$x^2+y^2+z^2+2xyz=1.$$

Solution. $\cos^{-1}x+\cos^{-1}y=\pi-\cos^{-1}z$;

$$\therefore\ \cos^{-1}\{xy-\sqrt{(1-x^2)(1-y^2)}\}=\pi-\cos^{-1}z.$$

$$\therefore\ xy-\sqrt{(1-x^2)(1-y^2)}=-\cos(\cos^{-1}z)$$

$$=-z$$

Hence $$xy+z=\sqrt{(1-x^2)(1-y^2)},$$
or $$x^2y^2+2xyz+z^2=1-x^2-y^2+x^2y^2.$$
$$\therefore \quad x^2+y^2+z^2+2xyz=1.$$

8·43. Prove that

$$\tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}}+\tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}}+\tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}=\pi.$$

Solution. Put $a+b+c=s$; the expression on the left hand side

$$=\tan^{-1}a\sqrt{\frac{s}{abc}}+\tan^{-1}b\sqrt{\frac{s}{abc}}+\tan^{-1}c\sqrt{\frac{s}{abc}}$$

$$=\tan^{-1}\frac{a\sqrt{\frac{s}{abc}}+b\sqrt{\frac{s}{abc}}+c\sqrt{\frac{s}{abc}}-abc\sqrt{\frac{s^3}{a^3b^3c^3}}}{1-bc\frac{s}{abc}-ca\frac{s}{abc}-ab\frac{s}{abc}}$$

$$=\tan^{-1}\frac{\sqrt{\frac{s}{abc}}\cdot s-\sqrt{\frac{s}{abc}}\cdot s}{1-s\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}=\tan^{-1}0=0 \text{ or } \pi.$$

The value zero is inadmissible as the sum of three positive angles cannot be zero.

8·44. Solve the equation

$$\cot^{-1}\frac{1}{x+1}+\cot^{-1}\frac{1}{x-1}=\tan^{-1}3x-\tan^{-1}x.$$

Solution. We have

$$\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}3x-\tan^{-1}x,$$

or $$\tan^{-1}\frac{(x+1)+(x-1)}{1-(x^2-1)}=\tan^{-1}\frac{3x-x}{1+3x^2},$$

or $$\tan^{-1}\frac{2x}{2-x^2}=\tan^{-1}\frac{2x}{1+3x^2},$$

$$\therefore \quad \frac{2x}{1+3x^2}=\frac{2x}{2-x^2}, \quad \therefore \quad x=0 \text{ or } 2-x^2=1+3x^2, \quad i.e., \quad x=\pm\tfrac{1}{2}.$$

8·45. *If* $xy=a^2+1$, *prove that*

$$\tan^{-1}\frac{1}{a+x}+\tan^{-1}\frac{1}{a+y}=\tan^{-1}\frac{1}{a}.$$

Deduce that $5\tan^{-1}\frac{1}{8}+2\tan^{-1}\frac{1}{18}+3\tan^{-1}\frac{1}{57}=\frac{\pi}{4}$.

Solution. $\tan^{-1}\frac{1}{a+x}+\tan^{-1}\frac{1}{a+y}=\tan^{-1}\frac{\frac{1}{a+x}+\frac{1}{a+y}}{1-\frac{1}{(a+x)(a+y)}}$

$=\tan^{-1}\frac{2a+x+y}{(a+x)(a+y)-1}=\tan^{-1}\frac{2a+x+y}{a^2+a(x+y)+xy-1}$

$=\tan^{-1}\frac{1}{a}$. $(\because\ xy-1=a^2)$.

Now $5\tan^{-1}\frac{1}{8}+2\tan^{-1}\frac{1}{18}+3\tan^{-1}\frac{1}{57}$

$=2\tan^{-1}\frac{1}{8}+2\tan^{-1}\frac{1}{18}+3\left(\tan^{-1}\frac{1}{7+1}+\tan^{-1}\frac{1}{7+50}\right)$

$=2\tan^{-1}\frac{1}{8}+2\tan^{-1}\frac{1}{18}+3\tan^{-1}\frac{1}{7}$

$[a=7,\ x=1,\ y=50,$ then $xy-1=a^2]$

$=2\tan^{-1}\frac{1}{8}+\tan^{-1}\frac{1}{7}+2\left(\tan^{-1}\frac{1}{5+13}+\tan^{-1}\frac{1}{5+2}\right)$

$=2\tan^{-1}\frac{1}{8}+\tan^{-1}\frac{1}{7}+2\tan^{-1}\frac{1}{5}[\because a=5,\ x=13,\ y=2,\ xy-1=a^2]$

$=2\left(\tan^{-1}\frac{1}{5+3}+\tan^{-1}\frac{1}{3+2}\right)+\tan^{-1}\frac{1}{7}$

$(\because\ a=3,\ x=5,\ y=2,\ xy-1=a^2)$;

$=2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{1}{3}+\left(\tan^{-1}\frac{1}{2+1}+\tan^{-1}\frac{1}{2+5}\right)$

$=\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{2}=\tan^{-1}\frac{1}{2+1}+\tan^{-1}\frac{1}{1+1}$, $(xy-1=a^2)$

$=\tan^{-1}\frac{1}{a}=\frac{\pi}{4}$.

Examples

Prove the following

1. $\tan^{-1}\frac{5}{12}=2\tan^{-1}5$.
2. $2\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{3}{4}$.
3. $2\tan^{-1}\frac{1}{3}=\cos^{-1}\frac{4}{5}$.
4. Find the value of $\cos(\sin^{-1}a+\cos^{-1}a)$.
5. Find the value of $\sin(\sin^{-1}a+\cos^{-1}a)$.
6. Find the value of $\tan(\sin^{-1}a+\cot^{-1}a)$.

Prove the following formulæ :—

7. $\tan^{-1}a-\tan^{-1}b=\tan^{-1}\frac{a-b}{1+ab}$.
8. $2\tan^{-1}x=\cos^{-1}\frac{1-x^2}{1+x^2}$.
9. $2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}$.

10. $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)=2\tan^{-1}x.$ 11. $\cos^{-1}(2x^2-1)=2\cos^{-1}x.$

12. $\tan\left\{\frac{1}{2}\sin^{-1}\dfrac{2x}{1+x^2}+\frac{1}{2}\cos^{-1}\dfrac{1-x^2}{1+x^2}\right\}=\dfrac{2x}{1-x^2}.$ *(C.U. 1912)*

13. Shew that $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\dfrac{\pi}{4}.$

14. Shew that $4\tan^{-1}\frac{1}{5}=\tan^{-1}\frac{120}{119}.$

15. Shew that $\tan^{-1}\frac{1}{2}=\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}=\dfrac{\pi}{4}.$

16. Show that $\cos^{-1}\dfrac{9}{\sqrt{82}}+\operatorname{cosec}^{-1}\dfrac{\sqrt{41}}{4}=\dfrac{\pi}{4}.$

17. Prove that
$\tan^{-1}n+\cot^{-1}(n+1)=\tan^{-1}(n^2+n+1).$ *(C. U. 1935)*

18. Show that $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=\dfrac{\pi}{4}.$

19. Prove that $\sin^{-1}\frac{1}{3}+\sin^{-1}\dfrac{1}{3\sqrt{11}}+\sin^{-1}\dfrac{3}{\sqrt{11}}=\dfrac{\pi}{2}.$
(C. U. 1915)

20. Prove that $\sin^{-1}\dfrac{4}{5}+\sin^{-1}\dfrac{5}{13}+\sin^{-1}\dfrac{16}{65}=\dfrac{\pi}{2}.$ *(C. U. 1920)*

21. Prove that
$\tan^{-1}1+\tan^{-1}2+\tan^{-1}3=\pi=2(\tan^{-1}1+\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3})$
(C. U. 1921)

22. Solve the equation
$$\sin^{-1}2x-\sin^{-1}x\sqrt{3}=\sin^{-1}x.$$

Sol. Taking sine of both the sides,
$$2x\cos(\sin^{-1}x\sqrt{3})-x\sqrt{3}\cos(\sin^{-1}2x)=x,$$
or
$$2x\sqrt{1-3x^2}-x\sqrt{3}.\sqrt{1-4x^2}=x,$$
whence $x=0$, or $x=\pm\frac{1}{2}.$

Solve the equations :—

23. $\sin^{-1}x+\sin^{-1}\dfrac{x}{2}=\dfrac{\pi}{4}.$

24. $\sin^{-1}\dfrac{5}{x}+\sin^{-1}\dfrac{12}{x}=\dfrac{\pi}{2}.$ *(C. U. 1914)*

25. $\tan^{-1}2x+\tan^{-1}3x=\dfrac{\pi}{4}.$

26. $\sin^{-1}\dfrac{2a}{1+a^2}+\sin^{-1}\dfrac{2b}{1+b^2}=2\tan^{-1}x.$

27. If $\sin(\pi\cos\theta)=\cos(\pi\sin\theta)$, show that
$$\theta=\pm\tfrac{1}{2}\sin^{-1}\tfrac{3}{4}.$$

28. Prove that $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}$
$$=2\tan^{-1}\tfrac{1}{5}+\tan^{-1}\tfrac{1}{7}+2\tan^{-1}\tfrac{1}{8}=\frac{\pi}{4}. \quad (C.\ U.\ 1922)$$

29. Prove that $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$
$$=\tfrac{1}{2}\left(\sin^{-1}\tfrac{1}{3}+\sin^{-1}\frac{1}{3\sqrt{11}}+\sin^{-1}\frac{3}{\sqrt{11}}\right). \quad (C.\ U.\ 1923)$$

30. Solve $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}x=\frac{\pi}{4}.$

31. Show that
$$\tan(2\tan^{-1}a)=2\tan(\tan^{-1}a+\tan^{-1}a^3) \quad (C.\ U.\ 1932)$$

32. Show that $2\tan^{-1}\left(\sqrt{\dfrac{a-b}{a+b}}\tan\dfrac{x}{2}\right)=\cos^{-1}\dfrac{b+a\cos x}{a+b\cos x}.$

33. Show that $\tan\left(\dfrac{\pi}{4}+\dfrac{1}{2}\cos^{-1}\dfrac{a}{b}\right)+\tan\left(\dfrac{\pi}{4}-\dfrac{1}{2}\cos^{-1}\dfrac{a}{b}\right)=\dfrac{2b}{a}.$

34. Prove that
$$\tan(\tan^{-1}x+\tan^{-1}y+\tan^{-1}z)=\cot(\cot^{-1}x+\cot^{-1}y+\cot^{-1}z).$$

35. Show that $\tan^{-1}x-\tan^{-1}z=\tan^{-1}\dfrac{x-y}{1+xy}+\tan^{-1}\dfrac{y-z}{1+yz}.$

36. Show that $\cot^{-1}\dfrac{ab+1}{a-b}+\cot^{-1}\dfrac{bc+1}{b-c}+\cot^{-1}\dfrac{ca+1}{c-a}=0.$

37. If $\cos^{-1}\dfrac{x}{a}+\cos^{-1}\dfrac{y}{b}=\alpha$, show that
$$\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha.$$

CHAPTER IX

COMPLEX NUMBERS

9·1. Extension of the notion of number. The domain of real numbers fails to give a solution of equations of the type

$$x^2+1=0.$$

The necessity of generalization of statements like "every quadratic has exactly two roots," "every cubic has three roots," requires a further advance of the number system. "That it is *permissible* to introduce these more general numbers is based upon the *freedom of scientific thought* to choose its own objects; that it is *desirable* to introduce such numbers is shown by the *result.*" The desirability of the introduction of more general types of numbers would be seen from the previous chapters and also those which follow. The new type of numbers, that we shall introduce are called **complex numbers** (or **imaginary numbers**, not a happy name). In the text, the words, complex and imaginary, will be used in the same sense, though the name imaginary will be avoided as far as possible.

9·11. Ordered pair of numbers. Let a and b be two real numbers. From these two numbers we can form another number $[a, b]$, which we call the product of the numbers, so that in our notation

$$[a, b]=ab.$$

The number $[b, a]$ also equals ab. Thus

$$[a, b]=[b, a].$$

In this number-pair, the *order* of the constituent parts is of *no importance.* The number-pair, above defined, is *not ordered.*

Let us now form from the real numbers a and b, another number $[a, b]$, which we may denote by a/b. With this definition of number-pair,

$$[a, b]\neq[b, a], \text{ when } a\neq b.$$

In this number-pair the order of a and b has a material effect on the new number formed. Such a number-pair is called *ordered*, in which the *order of the constituent parts is of importance.*

As another illustration, suppose that two directed mutually perpendicular lines, $X'OX$, $Y'OY$ have been chosen as the axes of co-ordinates. The position of a point P is determined by two real numbers x and y, its distances from OY and OX. The point is represented by the number-pair (x, y). The points represented by (x, y) and (y, x) are, in general, different. These points will coincide only if $x=y$. The above shows

the number-pair (a, b) is called ordered if the number-pairs (a, b) and (b, a) are distinct, unless $a=b$.

9·12. Definition of complex numbers. *The complex numbar $z=x+iy$ is defined by the ordered pair, (x, y), of real numbers x and y.*

The student must bear in mind that so far we have attached no meaning to + and i, $x+iy$ being merely another method of writing the number-pair (x, y).

9·2. Geometrical representation of complex numbers.

Argand's Diagram. In Art. 1·12 attention was drawn to a (1, 1) correspondence between the aggregate of real numbers and the totality of points on a directed st. line, by which to each point of the line there corresponds a definite real number, called its *abscissa*, and to each number there corresponds a definite point called the *image of the number*. The above definition of complex numbers as an ordered pair of numbers would immediately suggest a geometrical representation of complex numbers which is due to *Gauss and Argand*.

The ordered pair of numbers, $z=(x, y)$, in the cartesian plane represents a definite point referred to a frame of reference consisting of a pair of mutually perpendicular directed lines X'OX, Y'OY, with a common zero-point. The complex number $z=(x, y)$ will be represented by a point of the Cartesian plane whose co-ordinates are (x, y), and conversely, to the point whose co-ordinates are x and y, there corresponds the complex number $z=(x, y)$, or $z=x+iy$ (see p. 250 for diagram).

We associate with each complex number $z=x+iy$, that point of the plane which has, with reference to a fixed rectangular system, the co-ordinates x and y, and conversely, to each point with the co-ordinates x, y, we associate the complex number $z=x+iy=(x, y)$.

In particular $o+io$ has origin as its image ; $x+io$, the point (x, o) on the **X**-axis ; and $o+iy$, the point (o, y) on **Y**-axis.

We have seen that there exists a (1, 1) correspondence between the complex numbers, and the points of the Cartesian plane. It follows therefore, that to each definite relation between the complex numbers, there corresponds a definite relation between the points of the Cartesian plane, and conversely. From each theorem involving complex numbers, we can deduce a theorem involving points of the Cartesian plane. Thus known geometrical theorems will furnish a powerful aid in developing the theory of complex numbers ; the proofs of course will be purely by analytical methods, geometrical language, with which the student is already familiar, being used merely for the sake of illustration and convenience.

9·21. Complex numbers and polar co-ordinates.

If (r, θ) be the polar co-ordinates of P (x, y), (see Fig. page 250) then
$$x=r\cos\theta, \quad y=r\sin\theta, \tag{1}$$
whose solution is
$$r=\sqrt{x^2+y^2}, \quad \theta=\tan^{-1}\frac{y}{x}. \tag{2}$$

The formulas (1) are correct in sign, if r is always taken as positive, and the positive direction of the angle is chosen to be counter clock-wise. Hence

every complex number can be written in the form
$$z=x+iy=r\,(\cos\theta+i\sin\theta),$$
where r is the positive square root,
$$\sqrt{x^2+y^2},$$
and is called the modulus of the complex number, $z=x+iy$, *and is denoted by* $r=|z|$.

The angle θ *is called the amplitude or argument of z, and is written as*
$$\theta=am\,z, \text{ or } \theta=arg\,z.$$

Now θ is a many-valued function of x and y, the values differing by 2π. The value of θ which lies between $-\pi$ and π, is called the **principal value of the amplitude.** *In what follows, when we speak of the amplitude of z, we shall always mean its principal value.*

It should be observed that the principal value of θ is not necessarily the principal value of $\tan^{-1}(y/x)$; For a given number $x+iy$, both $\cos\theta$ and $\sin\theta$ have given values, therefore θ has only one value which lies between $-\pi$ and π.

The amplitude of a positive real number is 0, that of a positive purely complex number is $\pi/2$, and of purely negative complex number is $-\pi/2$. The amplitude of a negative real number, as defined above, is ambiguous, $\pm\pi$. We shall, however, consider it to be π. It will be seen that if $x>0$, $\theta=\tan^{-1}(y/x)$; if $x<0<y$, $\theta=\pi+\tan^{-1}(y/x)$; if $x<0$, $y<0$ $\theta=-\pi+\tan^{-1}(y/x)$.

Ex. 1. Find the modulus and amplitude of

$$-\frac{1}{2}+i\frac{\sqrt{3}}{2}.$$

Here $-\frac{1}{2}=r\cos\theta$, $\frac{\sqrt{3}}{2}=r\sin\theta$.

It follows that $r=1$, and $\theta=\frac{2\pi}{3}$.

The point $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$, corresponding to the complex number, lies in the third quadrant.

Note. It would obviously be wrong to obtain θ from $\tan\theta=-\sqrt{3}$, for $\tan^{-1}(-\sqrt{3})=-\frac{\pi}{3}$.

Ex. 2. Find the modulus and amplitude in each of the following cases, and plot the corresponding points verifying them by their polar co-ordinates also :—

(*i*) $\frac{1}{2}-i\frac{\sqrt{3}}{2}$, (*ii*) $-\frac{1}{2}-i\frac{\sqrt{3}}{2}$,

(*iii*) $-1-i$, (*iv*) -2, (*v*) $-2i$.

9·22. Vectors. By a vector we mean a line of definite length, and definite direction drawn in a definite sense, the position being of no concern.

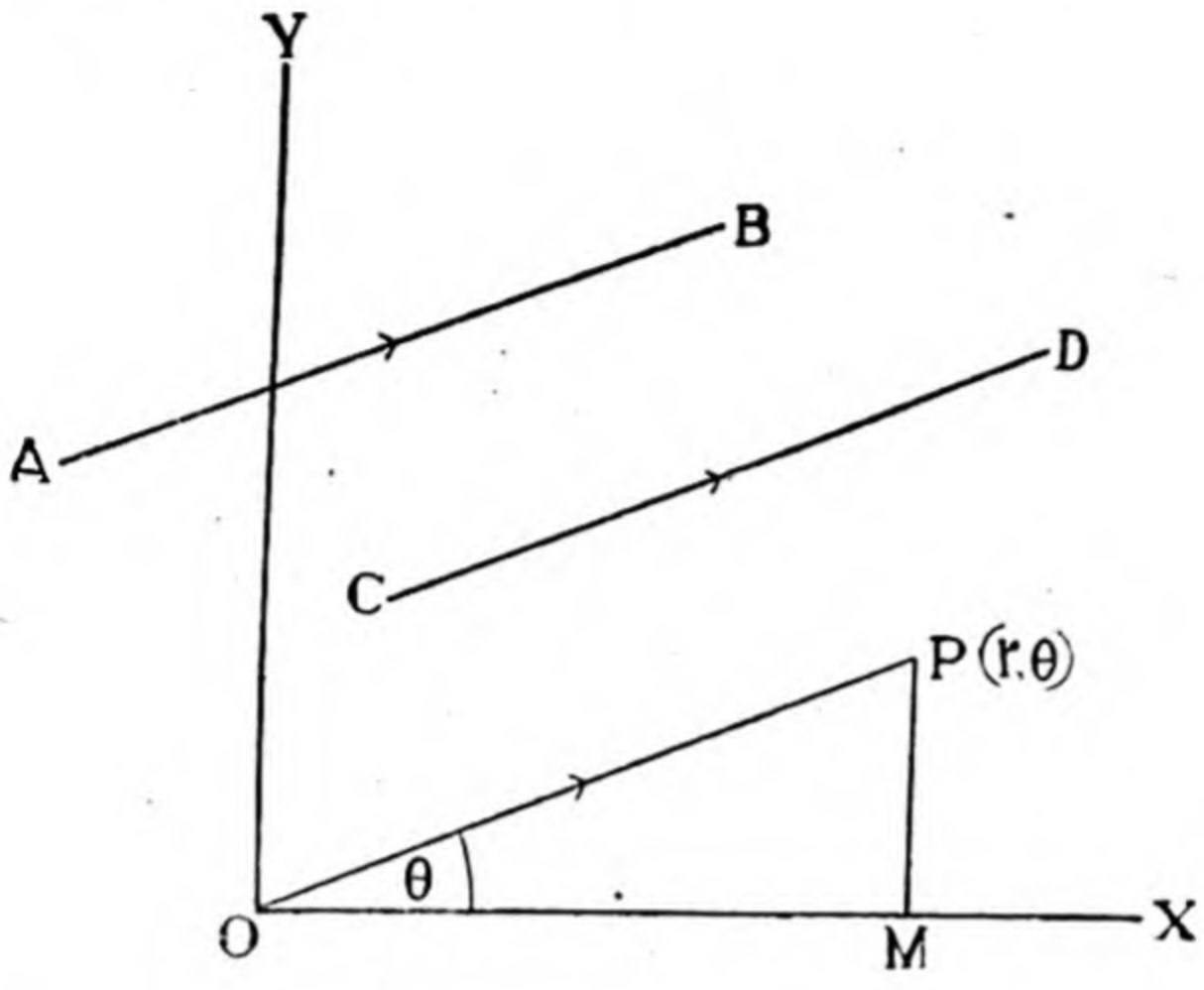
The vectors $\vec{AB}$, $\vec{CD}$ are said to be equal when they are of equal lengths and are drawn along parallel lines in the same sense.

It follows that the displacement of a point from A to B is a vector.

In comparing vectors, it is convenient, in general, to draw them from the same origin O.

Then each vector $\vec{OP}$ is terminated by its terminal point P. There is, therefore, (1, 1) correspondence between the vector

$\overrightarrow{OP}$ and the end point P, whose co-ordinates in a plane are represented by (x, y), to which corresponds a unique complex number $z=x+iy$.



Thus there is a (1, 1) reversible correspondence between the vector $\overrightarrow{OP}$, the terminal point P(x, y) and the complex number $z=x+iy$.

It may be noticed that the projection of the vector which represents the complex number $z=x+iy$ on X- and Y- axes, respectively, are x and y. The number x is called the real part of z, and y the imaginary part, and written as

$$x=\mathbf{R}(z), \quad y=\mathbf{I}(z).$$

The reason for this will be clear later.

9·3. Equality of complex numbers. *Two complex numbers* $z_1=x_1+iy_1$, *and* $z_2=x_2+iy_2$ *are equal, if, and only if,* $x_1=x_2$, $y_1=y_2$.

This definition is justified by the geometrical representation of the complex numbers; as the points (x_1, y_1), (x_2, y_2) in a Cartesian plane coincide only, if $x_1=x_2$, $y_1=y_2$.

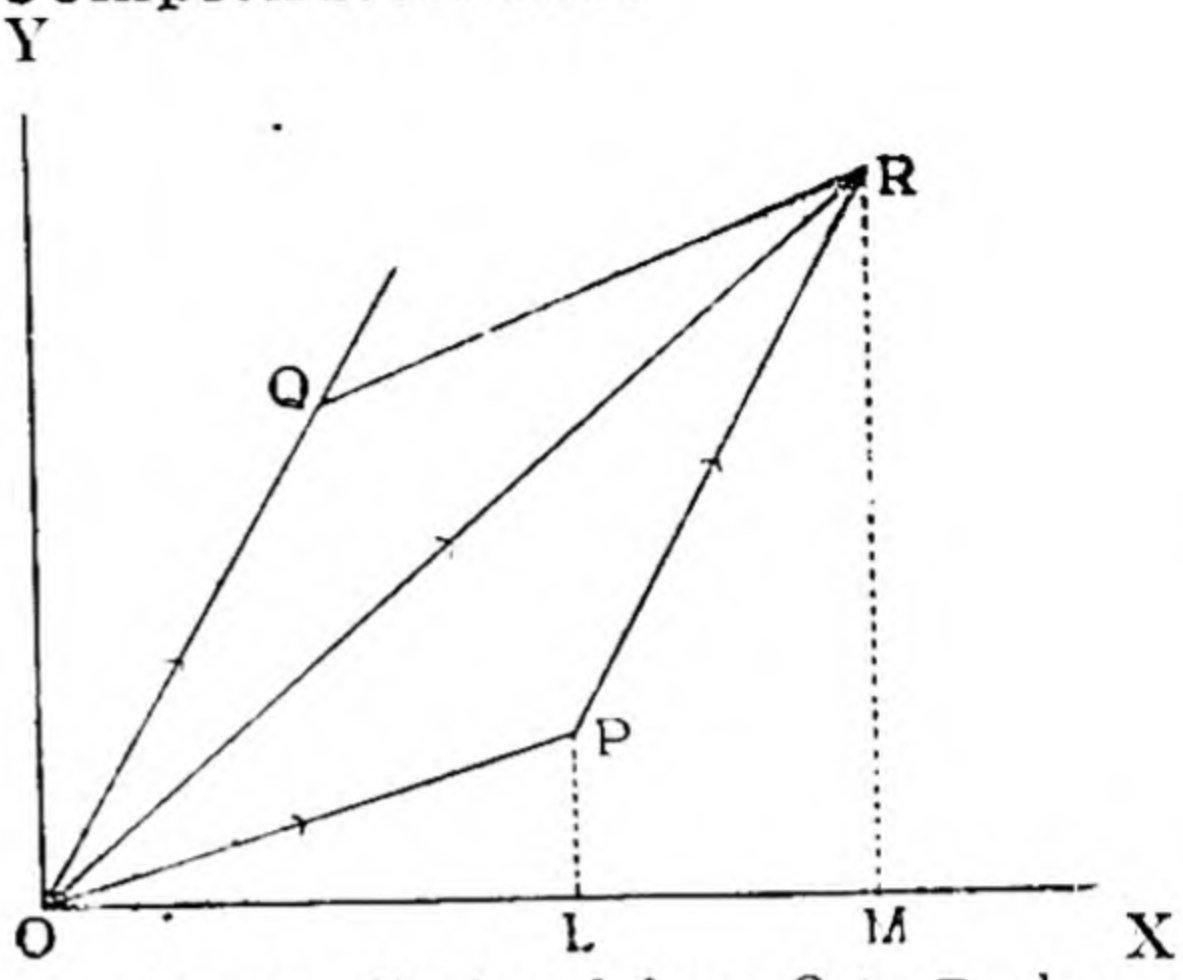
The above remark should not be regarded as a *proof*, but only a *suggestion* for the definition.

We may observe here that complex numbers with different amplitudes cannot be *ordered*, *i.e.*, given z_1 and z_2, we cannot establish either of the relations

$$z_1>z_2, \text{ or } z_1<z_2.$$

9·4. Addition of Complex Numbers.

Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ be the complex numbers represented by the displacements OP, OQ. Complete the ||grm. OPRQ. Then PR equals OQ and is drawn in the same sense. Thus the displacement PR is equal to OQ. If now a particle be displaced from O to P, and then from P to R, the final position will be the same as it were displaced from O to R along OR. Thus



$$\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OP} + \overrightarrow{OQ},$$

Now the projections of OP and PR on OX are x_1 and x_2 respectively, therefore, the projection of OR on OX is $x_1 + x_2$.

Similarly, the projection of $\overrightarrow{OR}$ on OY is $y_1 + y_2$. Hence the point R is $(x_1 + x_2,\ y_1 + y_2)$. Consequently, vector OR is $x_1 + x_2 + i(y_1 + y_2)$

Thus $$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2).$$

The above observations justify the definition which follows.

Definition (2) of addition. *If* $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, then the sum of z_1 and z_2 is defined by the relation

$$z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2) = (x_1 + x_2,\ y_1 + y_2).$$

Thus *the operation of addition of number-pairs (complex numbers) is definite, always possible and unique.*

9·41. From the above discussion it follows that

$$z = (x,\ y) = (x,\ 0) + (0,\ y)$$

so that (x, y) is the sum of the number pairs, $(x, 0)$ and $(0, y)$.

The sum so defined obeys the *commutative* and *associative* laws, *viz.*,

$$z_1 + z_2 = z_2 + z_1,$$

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3).$$

To prove the first, we note that the left-hand side is equal to $x_1 + x_2 + i(y_1 + y_2)$, and the right-hand side is equal to

$$x_2 + x_1 + i(y_2 + y_1);$$

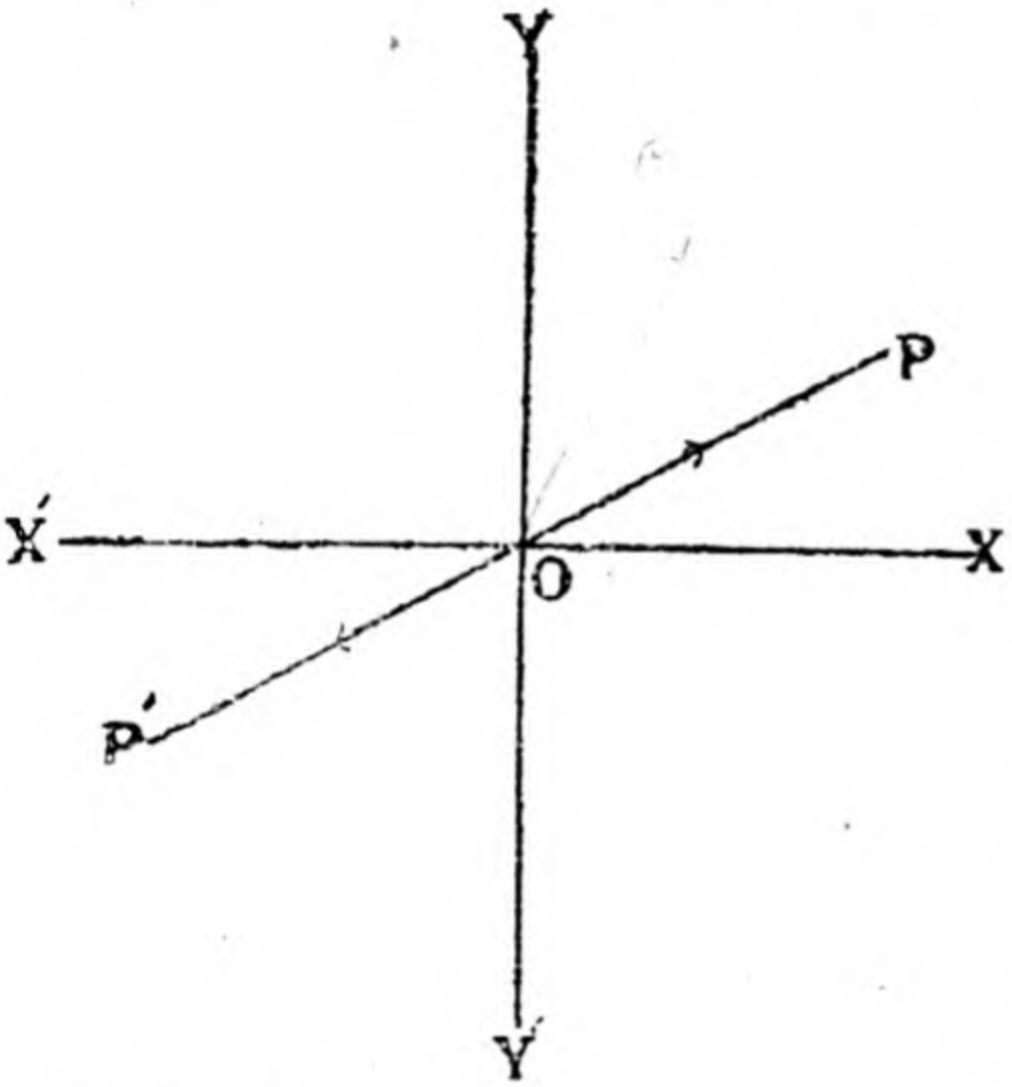
and these two are equal by definition of equality.

The second result follows similarly.

The following generalization of the above theorems can be easily deduced by following the line of proof as indicated above. The statement runs :

To sum any number of complex numbers, the separate summands may be combined into a smaller number of other summands by arbitrary selections and arrangements.

9·42. Subtraction of Complex Numbers.



Let $z_1=x_1+iy_1$ be represented by the point P. Suppose that the displacement $\overrightarrow{OP'}$ equals $\overrightarrow{OP}$, but is in the opposite direction, then obviously the point P′ is $(-x_1, -y_1)$, *i.e.*, $-x_1+i(-y_1)$. We may therefore define :—

Definition 3. ***If*** $z=x+iy$,, i.e. ***if*** (x, y) ***be a complex number, then*** $-z$ ***is defined*** as

$$z=-x+i(-y),$$

i.e., $(-x, -y)$.

Finally we define:

Definition 4.

$$z_1-z_2=z_1+(-z_2),$$

i.e.,

$$z_1-z_2=x_1-x_2+i(y_1-y_2),$$

i.e.,

$$(x_1-x_2, y_1-y_2).$$

Geometrically, if z_1, z_2, represent the vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$ respectively, z_1-z_2 represents the vector $\overrightarrow{QP}$, so that the vector which connects the points z_1 and z_2 in the Argand diagram, in the sense from Q to P, *i.e.*, z_2 to z_1, represents the number z_1-z_2 and its length is $|z_1-z_2|$,

Also $$am\ \overrightarrow{QP}=am\ (z_1-z_2),$$

Note. Any vector PQ may be referred to the origin by the rule

$$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP},$$

which is merely another way of writing the obvious truth

$$\overrightarrow{OP}+\overrightarrow{PQ}=\overrightarrow{OQ}.$$

9·5. Multiplication and Division of a number-pair and a real number.

Multiplication of a number-pair and a real number.

Let $z=(x, y)$ be the number-pair (*i.e.*, complex number) and m a real number.

If m is a positive integer, then the product $m\,(x, y)$ is the sum of m equal number-pairs (x, y).

This definition easily gives

$$m\,(x, y)=(x, y)+(x, y)+\ldots\ldots \text{ } m \text{ summands}$$
$$=(mx, my).$$

$\therefore$ $m(x, y)=(mx, my)$, if m is *a* positive integer.

In case m is a negative integer, then also the equation

$$m(x, y)=(mx, my)$$

will be taken as the definition of the product.

9·51. Division. *The quotient of a number-pair (x, y) by an integer m is the number-pair which when multiplied by m gives (x, y).*

Let $$\frac{1}{m}(x, y)=(a, b),$$

$\therefore$ by definition $$(x, y)=m(a, b)$$
$$=(ma, mb),$$

$\therefore$ $x=ma$, $y=mb$, [definition of equality

or $a=x/m$, $b=y/m$.

Hence $$\frac{1}{m}\,(x, y)=\left(\frac{x}{m}, \frac{y}{m}\right).$$

9·52. Multiplication by a fractional number.

The product of a rational number p/q and a number-pair $z=(x, y)$ is defined as the multiplication by p and division by q

i.e., $$\frac{p}{q}(x, y)=\left(\frac{px}{q}, \frac{py}{q}\right).$$

Finally the product of $z=(x, y)$ and a real number α, is defined by the equation $$\alpha(x,y)=(\alpha x, \alpha y)$$

We deduce the following theorem :—

9·53. Theorem. $(x, y)=x(1, 0)+y(0, 1)$

Taking the right-hand side,

$$x(1, 0)+y(0, 1)=(x, 0)+(0, y).$$
$$=(x, y).$$

Thus every number-pair can be expressed as the algebraic sum of the multiples of two special number-pairs $u_1=(1, 0)$, $u_2=(0, 1)$;

i.e., $$(x, y)=xu_1+yu_2.$$

Since $u_1=(1,\ 0)$ is the unit point on the x-axis in the Argand's diagram, $(1,\ 0)=1$; and if we denote u_2 by i, we have the common notation

$(x,\ y)=x+iy$; i so far denotes a unit only.

Examples IX (a)

Express in the form $r(\cos\theta+i\sin\theta)$ the following :

1. $1+\sqrt{-3}$. 2. $1+i$. 3. $-\sqrt{3}+i$. 4. $1+\sqrt{2}+i$.
5. $5+\sqrt{5}+\sqrt{-5}$. 6. $1+i\tan\alpha$. 7. $1+i\cot\alpha$.
8. $\tan\alpha-i$. 9. $1+\sin\theta+i\cos\theta$.

[We solve (9) as follows : Put $1+\sin\theta=r\cos\phi$, $\cos\theta=r\sin\phi$;

$$\therefore\ r^2=(1+\sin\theta)^2+\cos^2\theta=2+2\sin\theta=2\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^2.$$

$$\therefore\ r=\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)\sqrt{2}=2\sin\left(\frac{\theta}{2}+\frac{\pi}{4}\right).$$

$$\tan\phi=\frac{\cos\theta}{1+\sin\theta}=\frac{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}{\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)^2}=\frac{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}=\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}$$

$$=\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right),\quad\therefore\quad\phi=\frac{\pi}{4}-\frac{\theta}{2}\ \Big].$$

If α and β are real numbers, prove that

10. $\alpha(\beta x,\ \beta y)=\beta(\alpha x,\ \alpha y)=(\alpha\beta x,\ \alpha\beta y)$,
11. $(\alpha+\beta)(x,\ y)=\alpha(x,\ y)+\beta(x,\ y)$,
12. $\alpha[(x_1,\ y_1)+(x_2,\ y_2]=\alpha(x_1,\ y_1)+\alpha(x_2,\ y_2)$.
13. Find the equation of the circle whose centre is the point A (a), and radius r.

Solution. If P(z) be the moving point, $\overline{\mathrm{AP}}=z-a$ and the length of the vector $z-a$ is $|\ z-a\ |$. Hence the locus is

$$|\ z-a\ |=r.$$

14. Show that the equation of the ellipse whose foci are the points S(z_1), S$'(z_2)$, and $2a$ the length of the major axis, can be written in the form

$$|\ z-z_1\ |+|\ z-z_2\ |=2a.$$

Prove also that the eccentricity of the ellipse is

$$|\ z_1-z_2\ |/2a.$$

15. If P(z_1), Q(z_2) be two points in the Argand's diagram, the mid-point M of PQ is $(z_1+z_2)/2$.

[If P and Q are the extremities of the vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$ then

$\overrightarrow{OP}+\overrightarrow{PM}=\overrightarrow{OM}$, $\overrightarrow{OQ}+\overrightarrow{QM}=\overrightarrow{OM}$. Adding

$\therefore \quad \overrightarrow{OP}+\overrightarrow{OQ}=2\overrightarrow{OM}, \quad \because \quad \overrightarrow{PM}=-\overrightarrow{QM}.$]

16. If M divides the join of $P(z_1)$, $Q(z_2)$ in the ratio $m_1 : m_2$, show that M is represented by

$$\frac{m_2z_1+m_1z_2}{m_1+m_2}.$$

17. Show that the centroid of the triangle formed by $P(z_1)$, $Q(z_2)$, $R(z_3)$ is

$$\frac{z_1+z_2+z_3}{3}.$$

18. If the line joining $P(1+2i)$, $Q(5-6i)$ is divided internally, and externally, at M and N, so that

$$\frac{PM}{MQ}=\frac{PN}{QN}=\frac{3}{1},$$

show that $\overrightarrow{MN}=3-6i$.

19. Show that the necessary and sufficient conditions that the three points $P(z_1)$, $Q(z_2)$, $R(z_3)$ may lie on a line are

$$az_1+bz_2+cz_3=0, \quad a+b+c=0,$$

where a, b, c are real numbers, not all zero.

9·6. Multiplication of two complex numbers.

Def. *If* $z_1=x_1+iy_1$, $z_2=x_2+iy_2$,

then

$$z_1z_2=x_1x_2-y_1y_2+i(x_1y_2+x_2y_1).$$

This definition may be justified as follows :—

Let the polar co-ordinates of $P(x_1, y_1)$, $Q(x_2, y_2)$ be (r, α), (ρ,β).

Take OA=1 on OX.

On OQ construct a triangle directly similar to OAP,

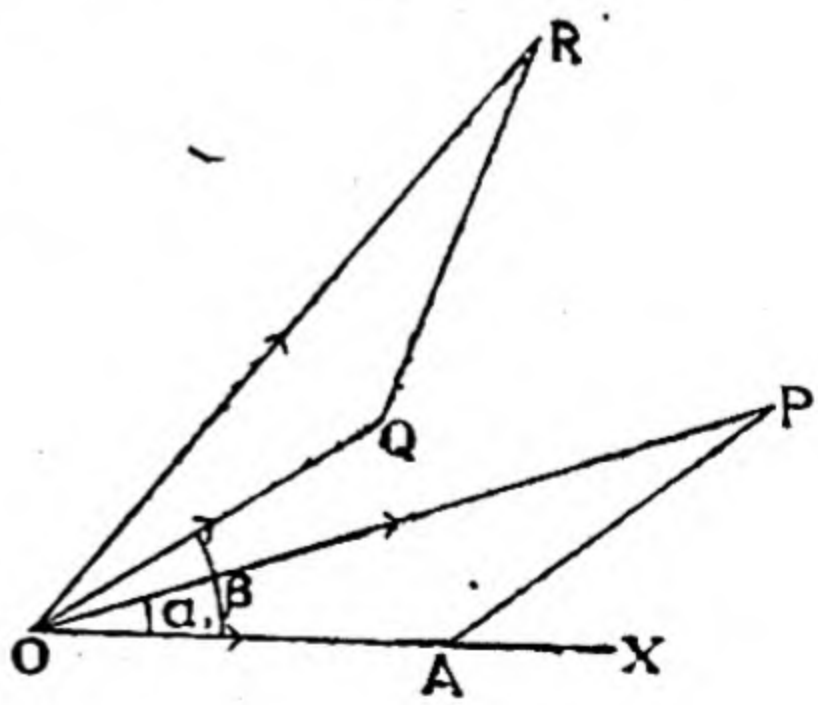
then $$\frac{OR}{OP}=\frac{OQ}{OA},$$

or $$OR.OA=OP.OQ,$$

i.e., $$OR=r.\rho.$$

$$\therefore \quad \overrightarrow{OR}=\overrightarrow{OP}\,.\,\overrightarrow{OQ},$$

and $$\angle AOR=\alpha+\beta.$$



Thus the Cartesian co-ordinates of R are

$x=r\rho\cos(\alpha+\beta)=r\rho(\cos\alpha\cos\beta-\sin\alpha\sin\beta)=x_1x_2-y_1y_2,$

$y=r\rho\sin(\alpha+\beta)=r\rho(\sin\alpha\cos\beta+\cos\alpha\sin\beta)=x_1y_2+x_2y_1.$

This justifies the definition.

Note that $$|z_1z_2|=|z_1|\,|z_2|,$$
$$am(z_1z_2)=am\,z_1+am\,z_2.$$

It should be observed that if α and β are the principlal values of the arguments of z_1 and z_2, then $\alpha+\beta$ is not necessarily the principal value of z_1z_2.

Note. We can not say *directly*, that

$$z_1z_2=r\rho(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)$$
$$=r\rho\{\cos(\alpha+\beta)+i\sin(\alpha+\beta)\}.$$

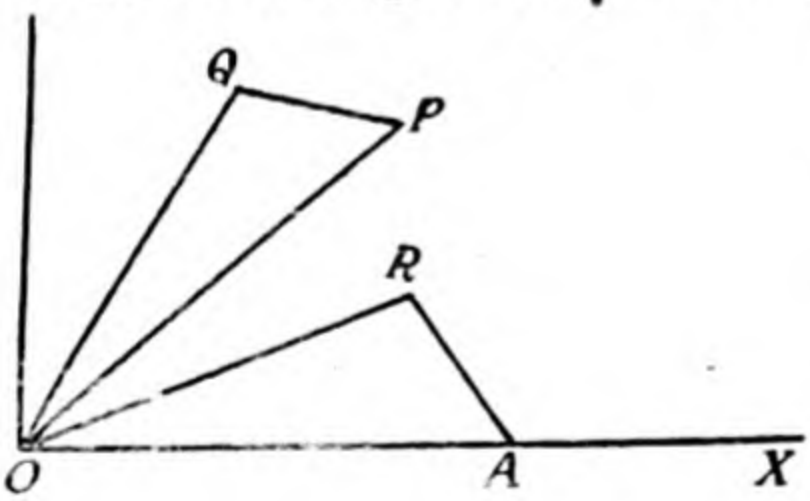
We have seen above that complex numbers obey the fundamental laws of algebra of real numbers. Hence the algebra of complex number will be identical in form with the algebra of real numbers. The significance of the two algebras, however, is different.

9.61. Division of complex numbers.

The following geometrical considerations lead to the definition of the law of division as given below.

Let $$z_1=x_1+iy_1=r(\cos\alpha+i\sin\alpha),$$
$$z_2=x_2+iy_2=\rho(\cos\beta+i\sin\beta),$$

be the points P and Q. Take OA=1 on OX, and on OA construct a triangle OAR directly similar to the triangle OPQ,



$$\therefore\ \frac{OQ}{OP}=\frac{OR}{OA}=OR;$$

and $$\angle AOR=\angle POQ=\beta-\alpha.$$

Thus the polar co-ordinates of R are

$$R\left(\frac{\rho}{r},\ \beta-\alpha\right).$$

If the Cartesian co-ordinates be (x, y)

$$x=\frac{\rho}{r}\cos(\beta-\alpha)=\frac{\rho r}{r^2}(\cos\alpha\cos\beta+\sin\alpha\sin\beta)$$
$$=\frac{x_1x_2+y_1y_2}{x_1^2+y_1^2},\ \because r^2=x_1^2+y_1^2.$$
$$y=\frac{\rho}{r}\sin(\beta-\alpha)=\frac{r\rho}{r^2}(\sin\beta\cos\alpha-\cos\beta\sin\alpha)$$
$$=\frac{x_1y_2-x_2y_1}{x_1^2+y_1^2}.$$

We accordingly define :

Definition. *The quotient z_2/z_1 of the complex numbers*

$$z_2=(x_2+iy_2)=(x_2, y_2),\ z_1=x_1+iy_1=(x_1, y_1),$$

is the number $$z=x+iy=(x, y),$$

where $$x=\frac{x_1x_2+y_1y_2}{x_1^2+y_1^2},\quad y=\frac{x_1y_2-x_2y_1}{x_1^2+y_1^2}.$$

In particular, if we agree to write as before, 1 for $1+i0$, then from above, if $z_2=(1, 0)$,

$$\frac{1}{z_1}=\left(\frac{x_1}{x_1^2+y_1^2},\ \frac{-y_1}{x_1^2+y_1^2}\right),$$

$$i.e.,\ \frac{x_1+i(-y_1)}{x_1^2+y_1^2}.$$

This defines the reciprocal of z_1.

It is to be noticed that

$$\left|\frac{z_2}{z_1}\right|=\frac{\rho}{r},\ am\ \frac{z_2}{z_1}=\beta-\alpha=am\ z_2-am\ z_1$$

Note. It may be observed, as in the case of the product, that $\beta-\alpha$ is not necessarily the principal value of $am\ (z_2/z_1)$.

9·611. It may be noticed that if P_1, P_2, P_3, are the points z_1, z_2, z_3 respectively,

$$am\ \frac{z_3-z_2}{z_1-z_2}\equiv\angle P_1\ P_2\ P_3.$$

9·62. The equation $i^2=-1$. We now try to simplify our notation so that the similarity between the forms of the algebras of complex and real numbers be closer. To this end we denote $x+i0$ by x, and $0+iy$, by iy, and in particular, $1.i$ by i, which is a number which corresponds to a unit displacement along the Y-axis.

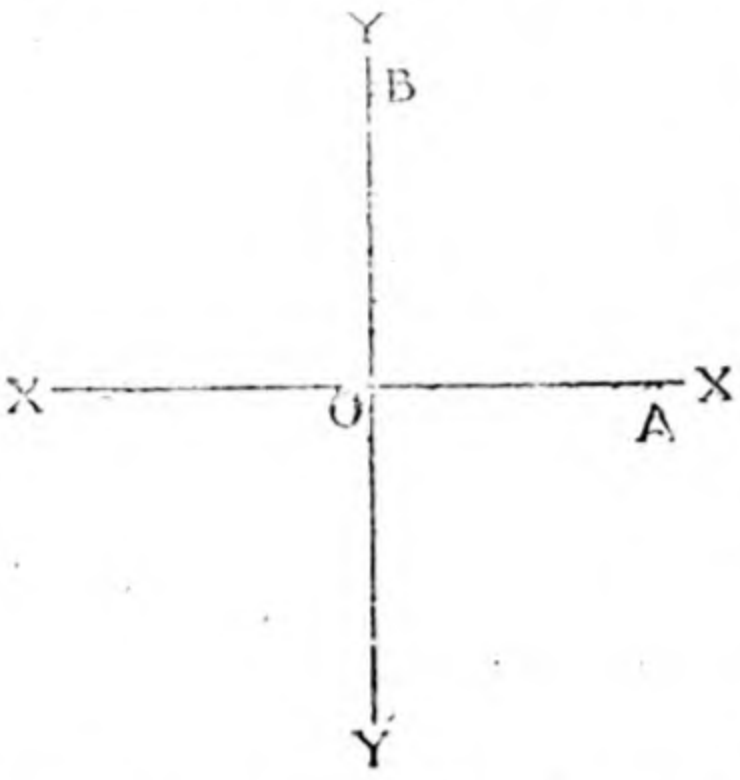
$i^2=i.i=(0+1i)(0+1i)=(-1, 0)=-1.$

Similarly $(-i)^2=-1.$

Thus i and $-i$ are the roots of the equation $x^2+1=0.$

With this convention, it is then easy to see that we can *operate with the camplex numbers in the same way as we operate with real numbers, regarding it as a number such that $i^2=-1$, wherever it occurs.*

Geometrical Significance of i. Here we may add the geometrical significance of i.



Suppose a unit displacement OA along x-axis is rotated through $+\frac{\pi}{2}$, so that it becomes the unit displacement along OY *i.e.*, $1.i=i$. Thus the effect of multiplying by i is equivalent to rotating a displacement through $+\frac{\pi}{2}$.

9·7. Conjugate Complex Numbers. If $z=x+iy$ be a complex number, $x-iy$ is defined to be the conjugate of z, and is denoted by $\bar{z}$. Geometrically, if a point P represants the complex number $z=x+iy=(x, y)$, the point Q which represents the conjugate number $z=x-iy=(x, -y)$, is the image of P in the real axis.

It is easily seen, that $|z|^2=z\bar{z}$, $z+\bar{z}=2\mathbf{R}(z)$, $z-\bar{z}=2i\mathbf{1}(z)$.

If z_1 and z_2 are complex numbers, and $\bar{z}_1$, $\bar{z}_2$, their conjugates, then the pairs

z_1+z_2 and $\bar{z}_1+\bar{z}_2$, $z_1+\bar{z}_2$ and $\bar{z}_1+z_2$; $z_1 z_2$ and $\bar{z}_1\bar{z}_2$; $z_1\bar{z}_2$ and $\bar{z}_1 z_2$

are sets of conjugate pairs.

9·8. We add some important theorems :

Theorem I. *The modulus of the product of two complex numbers is the product of their moduli.*

$$|z_1 z_2|^2=z_1 z_2 \bar{z}_1\bar{z}_2=z_1\bar{z}_1.z_2.\bar{z}_2=|z_1|^2.|z_2|^2.$$
$$\therefore \quad |z_1 z_2|=|z_1|\,|z_2|.$$

9·81. Theorem II. *The modulus of the sum of any number of complex numbers is at most equal to the sum of their moduli.*

If z_1, z_2...be the complex numbers,

$$|z_1+z_2+z_3+\ldots| \leqslant |z_1|+|z_2|+|z_3|+\cdots$$

We will prove the theorem for two complex numbers, the extension being immediate by induction. The result is evident, geometrically, from Fig. 9·4.

$$|z_1+z_2|=\mathrm{OR},\ |z_1|=\mathrm{OP},\ |z_2|=\mathrm{PR}.$$

and $$\mathrm{OR}<\mathrm{OP}+\mathrm{PR}.$$

The sign of equality will hold if P, Q, R are collinear.

9·82. The following is the *arithmetical proof.*—

Let $z_1=r_1(\cos\theta_1+i\sin\theta_1)$,

$z_2=r_2(\cos\theta_2+i\sin\theta_2)$

$$|z_1+z_2| = |(r_1\cos\theta_1+r_2\cos\theta_2)+i(r_1\sin\theta_1+r_2\sin\theta_2)|$$

$$=[(r_1\cos\theta_1+r_2\cos\theta_2)^2+(r_1\sin\theta_1+r_2\sin\theta_2)^2]^{\frac{1}{2}}$$

$$=[r_1^2+r_2^2+2r_1r_2\cos(\theta_1-\theta_2)]^{\frac{1}{2}}$$

$$\leqslant[r_1^2+r_2^2+2r_1r_2]^{\frac{1}{2}} \qquad [\because \cos(\theta_1-\theta_2)\leqslant 1$$

$$=r_1+r_2=|z_1|+|z_2|$$

$$\therefore\ |z_1+z_2|\leqslant|z_1|+|z_2|\,.$$

The equality will hold only when $\cos(\theta_1-\theta_2)=1$,

$$i.e.,\ \theta_1-\theta_2=2n\pi,$$

n being an integer.

9·83. Theorem III. *To prove* $|z_1-z_2|\geqslant||z_1|-|z_2|$.

Let $z_1=r_1(\cos\theta_1+i\sin\theta_1)$, $z_2=r_2(\cos\theta_2+i\sin\theta_2)$.

$$|z_1-z_2|^2=|(r_1\cos\theta_1-r_2\cos\theta_2)+i(r_1\sin\theta_1-r_2\sin\theta_2)|^2$$

$$=r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)\geqslant r_1^2+r_2^2-2r_1r_2=(r_1-r_2)^2.$$

$$[\because\ \cos(\theta_1-\theta_2)\leqslant 1]$$

$$\therefore\ |z_1-z_2|\geqslant||z_1|-|z_2||$$

The sign of equality will hold

$$\text{if } \theta_1-\theta_2=2n\pi,$$

i.e., when z_1, z_2 are collinear with the origin.

9·84. Theorem IV.

$$|z_1+z_2|^2+|z_1-z_2|^2=2\{|z_1|^2+|z_2|^2\}.$$

Proof. Let $z_1=r_1(\cos\theta_1+i\sin\theta_1)$, $z_2=r_2(\cos\theta_2+i\sin\theta_2)$.

$$\therefore\ |z_1+z_2|^2+|z_1-z_2|^2=\{r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)\}$$

$$+\{r_1^2+r_2^2+2r_1r_2\cos(\theta_1-\theta_2)\}$$

$$=2(r_1^2+r_2^2)=2|z_1|^2+2|z_2|^2.$$

This is the analytic equivalent of the geometrical theorem that if D be the mid-point of AB, then $OA^2+OB^2=2(OD^2+AD^2)$.

Examples IX (*b*)

Express the following in the form $A+iB$:—

1. $(1+i)^2$, $(1-i)^2$, $\left(\dfrac{1+i}{1-i}\right)^2$, $\left(\dfrac{a+ib}{a-ib}\right)$, $\left(\dfrac{a+ib}{a-ib}\right)^2$, $\dfrac{2-i}{(1-2i)^2}$, $\dfrac{5+i}{2+3i}$.

2. $\dfrac{(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)}{\cos\gamma-i\sin\gamma}$, $\dfrac{1+\cos2\theta-i\sin2\theta}{\cos2\theta+i\sin2\theta}$,

$$\frac{1-\cos2\theta+i\sin2\theta}{1+\cos2\theta-i\sin2\theta}.$$

3. Find the modulus and amplitude of

$$\frac{5+i\sqrt{3}}{4-2i\sqrt{3}}, (1+i)\frac{2+i}{3-i}.$$

4. If $c^2+s^2=1$, show that

$$\frac{1+c+is}{1+c-is}=c+is, \quad \frac{1+s+ic}{1+s-ic}=s+ic.$$

5. Find the modulus and amplitude of (*i*) the sum, and (*ii*) the product of the roots of the equation

$$(2+3i)z^2-(5+i)z+(7+4i)=0.$$

6. If $x+iy=\dfrac{p}{q+\cos\theta+i\sin\theta}$, show that the locus of (x, y), when θ is the parameter, is the circle

$$(q^2-1)(x^2+y^2)-2pqx+p^2=0,$$

p and q being real numbers.

7. Prove that the points $a+ib$, $1/(-a+ib)$ lie on a st. line through the origin.

8. Prove that the six points

$$1, -1, a+ib, -a+ib, 1/(a+ib), 1/(-a+ib)$$

on the Argand's diagram lie on a circle.

9. Show that the four points

$$-1, 1, 3+4i, \frac{1}{3+4i}$$

on the Argand's diagram lie on a circle of radius $\sqrt{10}$ and centre at the point $3i$.

10. In the parallelogram OPRQ, the side OP and diagonal OR represent, respectively, the numbers $5+i$ and $2+4i$. Find the complex numbers represented by the vectors OQ and RQ.

[See Fig. art. 9·4. $\vec{OP}+\vec{PR}=\vec{OR}$, $\therefore$ $\vec{PR}=(2+4i)-(5+i)$

$=-3+3i=\vec{OQ}$ $\quad RQ=-OP=-5-i$].

11. If a_1, a_2, a_3 are real numbers, show that the points which represent the complex numbers z_1, z_2, z_3 are collinear, if

$$(a_2-a_3)z_1+(a_3-a_1)z_2+(a_1-a_2)z_3=0.$$

12. If r_1, r_2 and θ_1, θ_2 are respectively, the moduli and principal values of the amplitudes, of two complex numbers, and r, θ the modulus and amplitude of their product, prove that if

(*i*) $\theta_1+\theta_2\leqslant-\pi$. $\quad \theta=\theta_1+\theta_2+2\pi$,

(*ii*) $-\pi<\theta_1+\theta_2\leqslant\pi$, $\quad \theta=\theta_1+\theta_2$,

(*iii*) $\pi<\theta_1+\theta_2$, $\quad \theta=\theta_1+\theta_2-2\pi$.

Examples IX (*c*)

1. If $z=x+iy$, express the following in the form $X+iY$, where X and Y are real functions of x and y :—

$$z^2,\ z^3,\ \frac{az+b}{cz+d};\ a,\ b,\ c,\ d \text{ being real numbers.}$$

2. If $a+b\sqrt{2}+(c+d\sqrt{2})i=A+B\sqrt{2}+(C+D\sqrt{2})i$, where a, b, c, d, A, B, C, D are real rational numbers.
then $$a=A,\ b=B,\ c=C,\ d=D.$$

3. Four points in the complex plane are represented by the complex numbers, α, β, γ, δ, in order, such that $\alpha+\gamma=\beta+\delta$, show that the four points are the corners of a parallelogram. (*P.U.1943*)

4. Show that the complex numbers $5+2i$, $7+5i$, $4+7i$, $2+4i$ are the vertices of a parallelogram. Find the angles and the lengths of the sides of the parallelogram.

5. Two opposite vertices of a square are, respectively, the complex numbers $9+12i$, $-5+10i$. Find the numbers which represent the other two vertices.

Let B be represented by $\alpha+i\beta$, and O be the origin (not shown in the fig :), then

$$\vec{BC}=\vec{OC}-\vec{OB}=9-\alpha+i(12-\beta),\ \vec{AB}=\alpha+5+i(\beta-10).$$

Now the vector $\vec{BC}$ is the vector $\vec{AB}$ rotated through $+\pi/2$.

$$\therefore\ 9-\alpha+i(12-\beta)=i[(\alpha+5(+i(\beta-10)],$$

$$\therefore\ 9-\alpha=10-\beta,\ 12-\beta=\alpha+5,$$

or, $$\alpha-\beta=-1,\ \alpha+\beta=7.$$

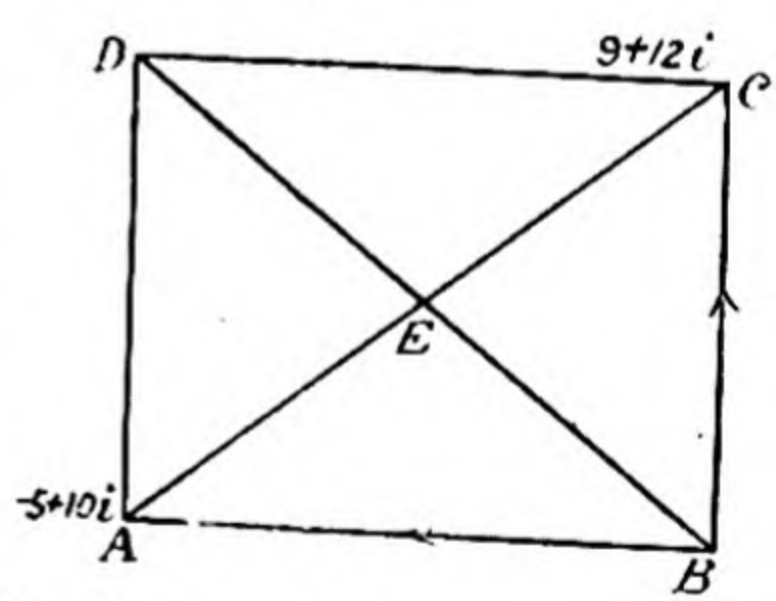
$$\therefore\ \alpha=3,\ \beta=4.$$

Thus B is represented by $3+4i$.

The point D may similarly be found ; or thus :

The mid-point E is represented by

$$\frac{-5+10i+9+12i}{2}=2+11i.$$



If D be the point $\gamma+i\delta$, $\dfrac{\gamma+i\delta+3+4i}{2}=2+11i$,

$$\therefore\ \gamma=1,\ \delta=18.$$

Thus D is represented by $1+18i$.

6. Two of the opposite vertices of a square are represented by the complex numbers $1-4i$ and $-1+10i$, show that the other pair of opposite vertices is represented by $7+4i$ and $-7+2i$.

7. If a pair of opposite vertices of a square be represented by the numbers $2+i$, $4+3i$, show that the other pair is represented by $4+i$, $2+3i$.

8. Show that the numbers $2+3i$, $8+11i$, $17i$ represent three points in Argand's diagram, which form a right-angled isosceles triangle. Find the number representing the fourth point which with the given three points would form a square.

9. If z_1, z_2, z_3 are the vertices of an isosceles triangle, right angled at the vertex z_2, prove that

$$z_1^2+z_3^2=2z_2(z_1+z_3-z_2).$$

What will be the co-ordinates of the fourth point which with these three would form a square.

[If A, B, C be the vertices, $\overrightarrow{AB}=\pm i\overrightarrow{BC}$.

$\therefore \quad z_1-z_2=\pm i\,(z_3-z_2)$].

10. If z_1, z_2, z_3, z_4 are the complex numbers which represent the vertices A, B, C, D, of a square in Argand's diagram, deduce that

$$2z_2=(1+i)z_1+(1-i)z_3,\ 2z_4=(1-i)z_1+(1+i)z_3.$$

11. Show that the points -1, $+1$, $i\sqrt{3}$ are the vertices of an equilateral triangle.

[If we call these points A, B, C; $\overrightarrow{AB}=2$, $\overrightarrow{BC}=-1+i\sqrt{3}$.

Now let the vector $\overrightarrow{AB}$ be rotated through $2\pi/3$, which is equivalent to multiplying the vector $\overrightarrow{AB}$ by Cis $2\pi/3$.

Thus we get

$$\overrightarrow{AB}\left(\cos\frac{2\pi}{3}+i\ \sin\frac{2\pi}{3}\right)=2\left(-\tfrac{1}{2}+\frac{i\sqrt{3}}{2}\right)=-1+i\sqrt{3}=\overrightarrow{BC}.$$

Similarly we can prove that $\overrightarrow{AC}=\overrightarrow{AB}$ Cis $\pi/3$.

1.2 Show that the points in the complex plane representing the numbers z, $z\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$, $z\left(\cos\frac{4\pi}{3}+i\ \sin\frac{4\pi}{3}\right)$, are the vertices of an equilateral triangle.

13. Show that, if the points z_1, z_2, z_3 are the vertices of an equilateral triangle, in the complex plane, then

$$z_1^2+z_2^2+z_3^2=z_2z_3+z_3z_1+z_1z_2.$$

Solution. If X, Y, Z be the vertices of the triangle,

$$\overrightarrow{ZX}=\overrightarrow{YZ}\ \text{cis}\frac{2\pi}{3}.$$

$$\therefore \quad z_1-z_3=(z_3-z_2)\left(-\tfrac{1}{2}+\frac{i\sqrt{3}}{2}\right),$$

or $\quad (2z_1-z_2-z_3)=i\sqrt{3}(z_3-z_2)$, or $(2z_1-z_2-z_3)^2+3(z_3-z_2)^2=0$,

which gives the result. The condition may be expressed in the form

$$(z_3-z_2)^2+(z_1-z_3)^2+(z_2-z_1)^2=0$$

or $$(\text{vector YZ})^2+(\text{vector ZX})^2+(\text{vector XY})^2=0.$$

14. Show that the above condition may be written as

$$\frac{1}{z_2-z_3}+\frac{1}{z_3-z_1}+\frac{1}{z_1-z_2}=0.$$

15. Two triangles whose vertices are a, b, c, and x, y, z respectively, will be similar, if

$(b-a)(z-x)=(c-a)(y-x)$, *i.e.*, $a(y-z)+b(z-x)+c(x-y)=0$.

Solution. If ABC, XYZ be the corresponding vertices of the triangles,

$$\frac{\overline{AB}}{\overline{AC}}=\frac{\overline{XY}}{\overline{XZ}} \quad i.e., \quad \frac{b-a}{c-a}=\frac{y-x}{z-x} \text{ etc.}$$

16. If $(x-y)(a-b)=(y-z)(b-c)=(z-x)(c-a)$, show that the triangles whose vertices are x, y, z and a, b, c are equilateral.

17. If z_1, z_2, z_3, z_4 are the complex numbers which represent the points A, B, C, D in the complex plane, such that

$$am\frac{z_2-z_3}{z_1-z_3}=am\frac{z_2-z_4}{z_1-z_4},$$

show that the points B and D are on the same side of the line joining A and C, and that the four points are concyclic.

[The above condition is equivalent to $\angle ACB=\angle ADB$ and since they have the same sign they lie on the same side of AB. By Euclid's well-known theorm, the four points are concyclic.]

18. If $am\frac{(z_2-z_3)(z_1-z_4)}{(z_1-z_3)(z_2-z_4)}=\pi$, prove that the points z_1, z_2, z_1, z_4 are concyclic.

If $z=x+iy$, and

19. $|z+1|=|z-1|\sqrt{2}$, then $x^2+y^2-6x+1=0$.

20. $|2z-1|=|z-2|$, then $x^2+y^2=1$.

In each case the point z describes a circle. If A, B be the points -1, and 1, the first result gives the geometrical condition $\frac{AP}{BP}=\sqrt{2}$, where P is the point z. The locus of P is a circle by the Apollonius theorem.

21. Prove that $|a+\sqrt{(a^2-b^2)}|+|a-\sqrt{(a^2-b^2)}|$
$=|a+b|+|a-b|$.

Put $z_1=a+\sqrt{a^2-b^2}$, $z_2=a-\sqrt{(a^2-b^2)}$. By Theorem 4,

$|z_1|^2+|z_2|^2=\frac{1}{2}|z_1+z_2|^2+\frac{1}{2}|z_1-z_2|^2$
$=2|a|^2+2|a^2-b^2|$.

Also $|z_1z_2|=|a^2-(a^2-b^2)|=|b^2|$.

$$\therefore \quad \{ |z_1| + |z_2| \}^2 = |z_1|^2 + |z_2|^2 + 2|z_1 z_2|$$
$$= 2[|a|^2 + |b|^2 + |a^2 - b^2|]$$
$$= |a+b|^2 + |a-b|^2 + 2|a+b|\,|a-b|$$
$$= \{ |a+b| + |a-b| \}^2.$$

22. If z_1, z_2 are the roots of $az^2+2\beta z+\gamma=0$, then
$$|a|(|z_1| + |z_2|) = |\beta + \sqrt{(a\gamma)}| + |\beta - \sqrt{(a\gamma)}|.$$

23. Prove that the transformation $z=\dfrac{Z-1}{Z+1}$, transforms the circle $x^2+y^2=a^2$ into the circle $(X-1)^2+Y^2=a^2[(X+1)^2+Y^2]$.

24. The two lines joining the points $z=a$, $z=b$ and $z=c$, $z=d$ will be perpendicular, if

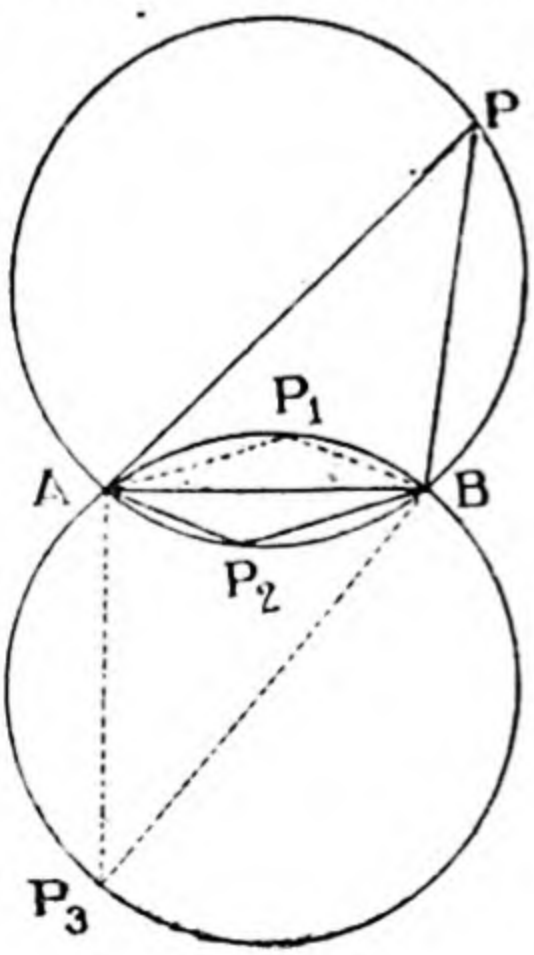
$am\left(\dfrac{a-b}{c-d}\right) = \pm\dfrac{\pi}{2}$, *i.e.*, $\dfrac{a-b}{c-d}$ is purely imaginary.

What is the condition that the lines may be parallel ?

25. Let (r, θ) denote the complex number whose modulus is r and amplitude θ. Then if $a\equiv(1, \alpha)$, $b\equiv(1, \beta)$, $c=(1, \gamma)$, and if $a+b+c=0$, prove that
$$a^{-1}+b^{-1}+c^{-1}=0.$$
Illustrate the result graphically. *(P. U. 1945)*

26. If z varies so that
$$am\,\frac{z-a}{z-b}=\theta,$$
where θ is a constant angle, then the point z describes an arc of a segment of a circle on ab, containing an angle θ.

[Let a, b, z be the numbers associated with A, B, P; A and B being fixed points, and P the variable point.



If the two circles in the figure be equal, and z_1, z_2, z_3 are the associate numbers of P_1, P_2, P_3, then obviously $\angle APB=\theta$. We thus see that
$$am\,\frac{z_1-b}{z_1-a}=\pi-\theta,\ am\,\frac{z_2-b}{z_2-a}=-\pi+\theta,$$
$$am\,\frac{z_3-b}{z_3-a}=-\theta.$$
Thus the locus of the equation
$$am\,\frac{z-b}{z-a}=\theta,$$
where θ is constant, is the arc APB. By writing $\pi-\theta$, $-\pi+\theta$, $-\theta$, respectively, for θ we obtain the arcs AP_1B, AP_2B, AP_3B. If θ is allowed to vary from $-\pi$ to π, we obtain

the system of coaxal circles of the intersecting species, A and B being the common points of the system. With each circle will be associated two supplementary values of θ.

27. Prove that if the amplitude of $\frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$ is zero, the four points representing z_1, z_2, z_3, z_4 in the Argand's Diagram lie on a circle or a straight line. (*P. U. 1915*)

28. If k is a constant, and z varies, so that

$$\left|\frac{z-a}{z-b}\right|=k,$$

then z describes a circle. If $k=1$, the locus is the right bisector of ab.

[If a, b, z be the associate numbers of A, B, P, the above equation is equivalent to the relation $PA/PB=k$. The locus of P by Apollonius theorem is a circle with A and B as inverse points. If k be allowed to vary, the equation represents a system of coaxal system of non-intersecting species with A and B as the limiting points.]

29. Show that the complex numbers z_1, z_2, z_3 represent three points on the complex plane which are concyclic with origin when

$$\frac{2}{z_2}=\frac{1}{z_1}+\frac{1}{z_3}.$$

30. If z_1, z_2, z_3, z_4 are four complex numbers, such that

$$\frac{z_1-z_4}{z_2-z_4}=k\,\frac{z_1-z_3}{z_2-z_3}.$$

show that the four points in the Argand's diagram represented by these numbers are concyclic.

31. The condition that the transformation

$$z=\frac{aZ+b}{cZ+d}$$

should make the circle $x^2+y^2=1$ correspond to a st. line in the X, Y plane is

$$a\,| = |\,c\,|\,.$$

CHAPTER X

RECIPROCAL EQUATIONS, D'MOIVRE'S THEOREM

10·1. Definition. *The equation*

$$f(x)\equiv p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots p_{n-2}x^2+p_{n-1}x+p_n=0$$

is called a reciprocal equation of the first class (kind), or if $p_s=p_{n-s}$ *for* $s=0, 1, 2,\ldots\ldots n$;
and of the second class (or kind) if

$$p_s=-p_{n-s} \text{ for } s=0, 1, 2,\ldots\ldots n.$$

A reciprocal equation of the first class and even degree is called a standard reciprocal equation.

The characteristic property of a reciprocal equation is the following :

If a *is any root of the equation, then* $\frac{1}{a}$ *is also a root.*

Proof. The equation whose roots are the reciprocals of the roots of the equation $f(x)=0$ is $x^n f\left(\frac{1}{x}\right)=0$. But

$$f(x)\equiv x^n f\left(\frac{1}{x}\right), \text{ or } f(x)=-x^n f\left(\frac{1}{x}\right),$$

according as the equation is a reciprocal equation of the first or the second class. Thus the roots of $x^n f\left(\frac{1}{x}\right)=0$ are exactly the roots of $f(x)=0$. This completes the proof.

10·11. *Every reciprocal equation can be reduced to a standard reciprocal equation.*

(1) *Let* $f(x)=0$ *be of the first class and odd degree, i.e.,* $n=2m+1$.

Since the roots of $f(x)=0$ occur in reciprocal pairs, one root will be its own reciprocal. This root is therefore ± 1. By actual substitution -1 is seen to be the root, and therefore $x+1$ is a factor of $f(x)$. Let.

$$f(x)=(x+1)\,\phi\,(x).$$

Then $\phi(x)$ is of even degree $n-1=2m$. Also

$$x^{2m}\phi\left(\frac{1}{x}\right)=x^{2m}\frac{f\left(\frac{1}{x}\right)}{\frac{1}{x}+1}=\frac{x^{2m+1}f\left(\frac{1}{x}\right)}{x+1}=\frac{f(x)}{x+1}=\phi(x).$$

Thus $\phi(x)=0$ is a reciprocal equation of the first class and even degree. $2m$.

(2) *Let $f(x)$ be of the second class.*

(*a*) If $n=2m+1$, arguing as in the previous case, $+1$ is seen to be a root of $f(x)=0$. If $f(x)=(x-1)\phi(x)$, then $\phi(x)$ is of degree $2m$, and

$$x^{2m}\phi\left(\frac{1}{x}\right)=x^{2m}\frac{f\left(\frac{1}{x}\right)}{\frac{1}{x}-1}=-\frac{x^{2m+1}f\left(\frac{1}{x}\right)}{x-1}=\frac{f(x)}{x-1}=\phi(x).$$

(*b*) If n is even and $=2m+2$, then $+1$ and -1 are both seen to be the roots of $f(x)$, so that $f(x)=(x^2-1)\phi(x)$, and $\phi(x)$ is of degrees $2m$.

$$x^{2m}\phi\left(\frac{1}{x}\right)=x^{2m}\frac{f\left(\frac{1}{x}\right)}{\frac{1}{x^2}-1}=\frac{-x^{2m+2}f\left(\frac{1}{x}\right)}{x^2-1}=\frac{f(x)}{x^2-1}=\phi(x).$$

Thus in every case the solution of $f(x)=0$ depends upon that of $\phi(x)=0$, which is of even degree and first class.

× 10.12. *A standard reciprocal equation is rationally reducible to an equation of half that degree.*

Let the equation be

$$p_0x^{2n}+p_1x^{2n-1}+\ldots+p_1x+p_0=0.$$

Dividing by x^n, the equation reduces to

$$p_0\left(x^n+\frac{1}{x^n}\right)+p_1\left(x^{n-1}+\frac{1}{x^{n-1}}\right)+\ldots+p_{n-1}\left(x+\frac{1}{x}\right)+p_n=0, \quad (1)$$

Put $x+\frac{1}{x}=z$, and let $x^s+\frac{1}{x^s}=V_s$, so that $V_1=z$, and x, $\frac{1}{x}$ are the roots of

$$t^2-tz+1=0. \quad (2)$$

$$\therefore\quad t^{s+2}-t^{s+1}z+t^s=0,$$

hence

$$x^{s+2}-x^{s+1}z+x^s=0,$$

and

$$\frac{1}{x^{s+2}}-\frac{1}{x^{s+1}}z+\frac{1}{x^s}=0,$$

for x, $\frac{1}{x}$ are the roots of the quadratic in t.

Addition gives

$$V_{s+2}-z\,V_{s+1}+V_s=0.$$

Also $$V_2=x^2+\frac{1}{x^2}=\left(x+\frac{1}{x}\right)^2-2=z^2-2.$$

Putting $s=1$, $V_3=z\,(z^2-2)-z=z^3-3z$,

and so on. Thus we express V_s as a polynomial of degree s. Substituting in (1) we get an equation of degree n. If we know any one root z_1 of this equation in z, the corresponding roots of $f(x)=$can be found from (2).

Examples X (*a*)

Solve the following equations :—

1. $2x^4+5x^3+4x^2+5x+2=0.$
2. $6x^5-x^4-43x^3+43x^2+x-6=0.$
3. $6x^6+5x^5-44x^4+44x^2-5x-6=0.$
4. $2x^8-x^7-12x^6+14x^5-14x^3+12x^2+x-2=0.$
5. $x^8+1+(x+1)^8=2(x^2+x+1)^4.$
6. $x^5-7x^4+x^3-x^2+7x-1=0.$
7. Show that the z-quartic for $x^9=1$ has the root -1. Hence prove that $x^9=1$ can be completely solved if the root of $z^3-3z+1=0$ be known.
8. Prove that the equation $x^7-1=0$ can be solved by the help of the cubic $x^3+x^2-2x-1=0$ which has real roots.
9. If $(x-1)^5=a(x^5-1)$, $x\neq1$, prove that

$$x+\frac{1}{x}=\frac{4+a\pm\sqrt{5a(4+a)}}{2(1-a)}.$$

Also if $0<a<16$, the given equation has three real and two imaginary roots, but if $a<0$, or $a>16$, the only real root is 1.

[**Solution.** If $z=x+x^{-1}$, then $(1-a)\,z^2-z(4+a)+4+a=0$.

Let z_1, z_2 be the roots of this equation, then

$x=\frac{1}{2}[z_1\pm\sqrt{z_1^2-4}]$, or $\frac{1}{2}\,[z_2\pm\sqrt{z_2^2-4}]$.

The given equation has obviously one real root 1. Among the remaining four roots, all are imaginary if z_1^2-4 and z_2^2-4 are both negative, and two real and two imaginary, if one of them is positive and the other negative.

The equation whose roots are z_1^2-4, z_2^2-4 is

$$(1-a)^2y^2+5a(a-6)y+5a(a-16)=0.$$

The roots of this equation have opposite signs if $a(a-16)<0$, $\therefore\ 0<a<16$. This proves the first part. If all the roots are imaginary, the roots of the above equation are both negative, $\therefore\ a(a-16)>0$ and $a(a-6)>0$.

The first condition gives either $a<0$, or $a>16$.
The second inequally is satisfied under these conditions.]

10. For the equation $x^4+px^3+qx^2+px+1=$, prove that
(*i*) the roots are all real if and only if
$$4(q-2)<p^2<(\tfrac{1}{2}q+1)^2 \text{ and } (2q+4)<p^2;$$
(*ii*) the roots are all imaginary if and only if
$p^2<(\frac{1}{2}q+1)^2$, and at least one of the two inequalities
$$p^2<4(q-2),\ p^2<(2q+4) \text{ holds};$$
(*iii*) two roots are real and two imaginary if $(\frac{1}{2}q+1)^2<p^2$.
[If $z_1=x+x^{-1}$, the equation reduces to $z^2+pz+q-2=0$.
If z, z_2 be the roots of this equation,
$$x=\tfrac{1}{2}[z_1\pm\sqrt{z_1^2-4}], \text{ or } \tfrac{1}{2}[z_2\pm\sqrt{z_2^2-4}].$$
If any value of x is real, one and, therefore, both values of z must be real; therefore $p^2>4(q-2)$.
Also z_1^2-4, z_2^2-4 are the roots of
$$y^2+y(2q+4-p^2)+(q+2)^2-4p^2=0 \text{ etc.}]$$

10·2. D'Moivre's Theorem. *For all real or complex values of n and θ,*

cos nθ+i sin nθ is a value of (cos θ+i sin θ)^n.

The theorem holds when
(1) θ and n are both real.
(2) θ complex and n real.
(3) θ real and n complex.
(4) θ and n both complex.
We shall only prove the first case under the restriction that n *is rational.*

The expression cos θ+i sin θ will be abbreviated into cis θ.

Case 1. *Let n be a positive integer.*

By actual multiplication
$$\text{cis } \theta_1.\ \text{cis } \theta_2=\text{cis}\,(\theta_1+\theta_2).$$
Again $\text{cis } \theta_1.\ \text{cis}\theta_2.\ \text{cis}\theta_3=\text{cis}\,(\theta_1+\theta_2)\ \text{cis}\ \theta_3=\text{cis}\,(\theta_1+\theta_2+\theta_3)$.
Proceeding in this way n times,
$$\text{cis } \theta_1\ \text{cis } \theta_2\ldots\text{cis } \theta_n=\text{cis}\,(\theta_1+\theta_2+\cdots+\theta_n). \quad \ldots (1)$$
Now suppose $\theta_1=\theta_2=\cdots=\theta_n=\theta$, then the above relation becomes
$$(\text{cis } \theta)^n=\text{cis } n\theta. \quad \ldots \quad \ldots \quad \ldots (2)$$

Case II. *When n is a negative integer.*

Suppose $n=-m$
$$(\text{cis } \theta)^{-n}=(\text{cis } \theta)^{+m}=\cos m\theta+i\sin m\theta$$
$$=\frac{1}{\cos m\theta-i\sin m\theta}=\frac{1}{\cos(-m\theta)+i\sin(-m\theta)}=\text{cis } n\theta.$$
$$\therefore\ (\text{cis } \theta)^n=\text{cis } n\theta.$$

Case III. *When n is a fraction, one value of $(\text{cis }\theta)^n$ is $\text{cis }n\theta$.*

Suppose $n=\frac{p}{q}$, when p and q are integers, and $q+ve$.

We have $\left(\text{cis }\frac{\theta}{q}\right)^q=\text{cis }\theta$. (By Case I)

$\therefore$ One value of $(\text{cis }\theta)^{\frac{1}{q}}$ is $\text{cis }\frac{\theta}{q}$.

Raising to the power p, we note that one value of

$$(\text{cis }\theta)^{\frac{p}{q}} \text{ is } \left(\text{cis }\frac{\theta}{q}\right)^p, \text{ } i.e., \text{ cis } \frac{p}{q}\theta. \quad [\text{By I or II.}]$$

This completes the proof for n being any rational number.

Examples X (*b*)

1. If $A+iB=(a_1+ib_1)(a_2+ib_2)\ldots(a_n+ib_n)$, prove that

$$\tan^{-1}\frac{B}{A}=\tan^{-1}\frac{b_1}{a_1}+\tan^{-1}\frac{b_2}{a_2}+\ldots+\tan^{-1}\frac{b_n}{a_n}.$$

[*P. U. 1941*]

[Put $A+iB=R(\cos\phi+i\sin\phi)$, $a_1+ib_1=r_1\text{ cis }\theta_1$,
$a_2+ib_2=r_2\text{ cis }\theta_2,\ldots a_n+ib_n=r_n\text{ cis }\theta_n$.
Then R cis $\phi=r_1r_2\ldots r_n$ cis θ_1 cis θ_2 cis $\theta_3\ldots$cis θ_n etc.]

2. If $2\cos\theta=x+\frac{1}{x}$ and $2\cos\phi=y+\frac{1}{y}$, show that one of the values of $\frac{x^m}{y^n}+\frac{y^n}{x^m}$ is $2\cos(m\theta-n\phi)$. [*P. U. 1937*]

3. Apply DeMoivre's theorem to find the cube roots of 1 and -1. [*P. U. 1945*]

4. If α, β be the roots of $x^2-2x+4=0$, prove that

$$\alpha^n+\beta^n=2^{n+1}\cos\frac{n\pi}{3}. \quad [\textit{P. U. 1941}]$$

10·21. To find the nth roots of a complex number.

Let the number be $a+ib$.

Suppose $a+ib=R(\cos\theta+i\sin\theta)$,

so that $a=R\cos\theta$, $b=R\sin\theta$,

$$\therefore\ R=(a^2+b^2)^{\frac{1}{2}},\ \tan\theta=\frac{b}{a}.$$

Consequently $(a+ib)^{\frac{1}{n}}=R^{\frac{1}{n}}(\cos\theta+i\sin\theta)^{\frac{1}{n}}$.

$=R^{\frac{1}{n}}\{\cos(\theta+2p\pi)+i\sin(\theta+2p\pi)\}^{\frac{1}{n}}$, where p is an integer.

Thus the n values of the nth root, *i.e.*, $(a+ib)^{\frac{1}{n}}$ are

$$R^{\frac{1}{n}}\left\{\cos\frac{\theta+2p\pi}{n}+i\sin\frac{\theta+2p\pi}{n}\right\}, \text{ where } p=0,\ 1,\ 2,\ldots\overline{n-1}$$

If p be given integeral values $>n-1$, these values are repeated, and no fresh root is obtained.

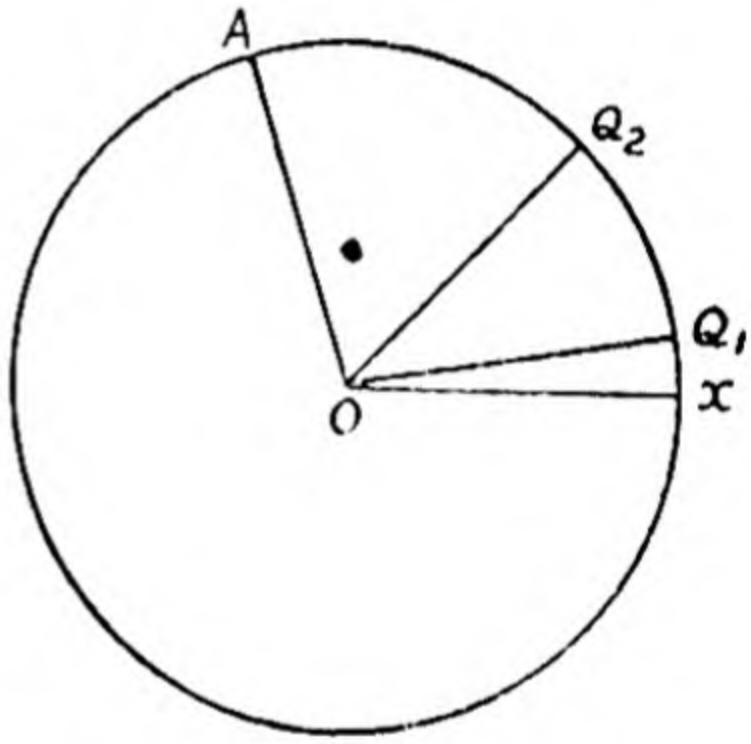
As already proved, the number of values cannot be greater than n. Also it is clear from the fact that these values are the roots of the equation $x^n=a+ib$, which cannot have more than n roots.

10·22. *The geometrical interpretation of the q values of* $(\cos\theta+i\sin\theta)^{\frac{1}{q}}$.

For the sake of simplicity, we shall take $q=12$, the method employed being quite general.

With O as centre describe a circle of unity radius. Let $\angle XOA=\theta$.

Let the line OP starting from OX revolve round O in the counter clock-wise direction. Every time that OP passes through OA, OA represents $\cos\theta+i\sin\theta$. Now let another line OQ start from OX and revolve with an angular velocity $\frac{1}{12}$th that of OP, then the position of OQ when OP passes through OA will represent the value of $(\cos\theta+i\sin\theta)^{\frac{1}{12}}$.



When OP passes through OA for the first time, OQ will be at OQ_1, such that $XOQ_1=\frac{\theta}{12}$. When OP passes through OA after revolving through $\theta+2\pi$, OQ shall have revolved through $\frac{\theta+2\pi}{12}$, and will take the position OQ_2, such that $\angle Q_1OQ_2=\frac{2\pi}{12}$. In this way, OQ will take positions OQ_3, OQ_4, etc.....OQ_{12} and the angle between every two successive positions of OQ will be $\frac{2\pi}{12}$; the points Q_1, Q_2 .....Q_{12} will thus form a regular polygon of 12 sides. This is similar to the minute hand of a clock passing every time through the figure XII as the hour-hand passes through I, II, etc.

10·23. The values of $(\cos\theta+i\sin\theta)^{p/q}$ where p and q are integers prime to each other and q is positive.

Let r be an integer positive or negative, then

$$\left\{\cos\left(\frac{p\theta+2\pi r}{q}\right)+i\sin\left(\frac{p\theta+2\pi r}{q}\right)\right\}^q$$
$$=\cos(p\theta+2\pi r)+i\sin(p\theta+2\pi r)$$
$$=\cos p\theta+i\sin p\theta=(\cos\theta+i\sin\theta)^p.$$

It follows, therefore, that $\operatorname{cis}\frac{p\theta+2\pi r}{q}$ is the general value of $(\cos\theta+i\sin\theta)^{p/q}$, where r is any integer, positive or negative.

Giving to r the values $0, 1, 2,\ldots q-1$, we see that each of the q values

$$\operatorname{cis}\frac{p\theta}{q},\ \operatorname{cis}\frac{p\theta+2\pi}{q},\ \operatorname{cis}\frac{p\theta+4\pi}{q},\ldots\ldots\operatorname{cis}\frac{p\theta+2\pi\overline{q-1}}{q},$$

is a value of $(\operatorname{cis}\theta)^{p/q}$.

These q values are all distinct. If the two values for $r=r_1$ and $r=r_2$ be equal,

$$\frac{p\theta+2\pi r_1}{q}-\frac{p\theta+2\pi r_2}{q},\ \text{or}\ \frac{2\pi(r_1-r_2)}{q}$$

must be a multiple of q. But since $r_1\leqslant q-1$, and also $r_2\leqslant q-1$, r_1-r_2 cannot be a multiple of q. Hence all the values are distinct.

Any value of r different from $0, 1,\ldots q-1$, will not give a root distinct from those in the above series.

For any such value of r would differ from one of the numbers $0, 1, 2,\ldots\overline{q-1}$, by a multiple of q.

Thus $(\operatorname{cis}\theta)^{p/q}$ has exactly q values given by

$$\operatorname{cis}\frac{p\theta+2\pi r}{q},\quad r=0, 1, 2,\ldots q-1.$$

Examples X (*c*)

Simplify the following

1. $\dfrac{(\cos\theta+i\sin\theta)^3(\cos\theta-i\sin\theta)^5}{(\cos 2\theta+i\sin 2\theta)^4}$.

2. $\dfrac{(\cos 3\theta+i\sin 3\theta)^5(\cos\theta-i\sin\theta)}{(\cos 5\theta+i\sin 5\theta)^4(\cos 2\theta-i\sin 2\theta)^3}$.

3. $\dfrac{(\cos 2\theta-i\sin 2\theta)^7(\cos 3\theta+i\sin 3\theta)^5}{(\cos 5\theta+i\sin 5\theta)^3(\cos 7\theta-i\sin 7\theta)^2}$.

4. $(1-\cos\theta+i\sin\theta)^8$. 5. $(1+\cos\theta+i\sin\theta)^{10}$.

Solve the equations

6. $(\cos\theta+i\sin\theta)(\cos 2\theta+i\sin 2\theta)(\cos 3\theta+i\sin 3\theta)=1$.

7. $(\cos\theta+i\sin\theta)(\cos 2\theta+i\sin 2\theta).....(\cos p\,\theta+i\sin p\theta)=1$.

8. If $(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)=1$, then $\alpha+\beta=2k\pi$, where k is an integer.

9. If $(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)=i$, then $\alpha+\beta=(4k+1)\pi/2$, where k is an integer.

10. If $x=\cos\alpha+i\sin\alpha$, show that

$$x+\frac{1}{x}=2\cos\alpha, \qquad x-\frac{1}{x}=2i\sin\alpha, \qquad \frac{x^2-1}{x^2+1}=i\tan\alpha,$$

$$x^n+\frac{1}{x^n}=2\cos n\alpha, \quad x^n-\frac{1}{x^n}=2i\sin n\alpha, \quad \frac{x^{2n}-1}{x^{2n}+1}=i\tan n\alpha.$$

11. If $2\cos\theta=x+\frac{1}{x}$ and $2\cos\phi=y+\frac{1}{y}$,

show that one of the values of

$$\frac{x^m}{y^n}+\frac{y^n}{x^m} \text{ is } 2\cos(m\theta-n\theta). \qquad (P.\ U.\ 1937)$$

12. Show that the cube roots of $1+i$ are

$$2^{1/6}\text{cis}\,\frac{\pi}{12},\ 2^{1/6}\,\text{cis}\,\frac{3\pi}{4},\ 2^{1/6}\text{cis}\,\frac{17\pi}{12}.$$

13. Find all the values of $(1-i)^{1/3}$.

14. Show that -1 is one of the values of

$$\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)^{\frac{11}{2}}\Big/\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^{1/2}.$$

15. Simplify

$\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)^{1/4}$, and express the results in a form free from trigonometrical expressions. *(P. U. 1945)*

16. If $\cos\alpha+\cos\beta+\cos\gamma=0$ and $\sin\alpha+\sin\beta+\sin\gamma=0$ prove that

$$\cos 3\alpha+\cos 3\beta+\cos 3\gamma=3\cos(\alpha+\beta+\gamma),$$
$$\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma).$$

Proof. Let $a=\text{cis}\,\alpha$, $b=\text{cis}\,\beta$, $c=\text{cis}\,\gamma$, then

$$a+b+c=0, \qquad \therefore\ a^3+b^3+c^3=3abc.$$

Now $a^3=\text{cis}\,3\alpha$, $b^3=\text{cis}\,3\beta$, $c^3=\text{cis}\,3\gamma$, $abc=\text{cis}\,(\alpha+\beta+\gamma)$,

$$\therefore (\cos 3\alpha+\cos 3\beta+\cos 3\gamma)+i(\sin 3\alpha+\sin 3\beta+\sin 3\gamma)$$
$$=3[\cos(\alpha+\beta+\gamma)+i\sin(\alpha+\beta+\gamma)].$$

Hence

$$\cos 3\alpha+\cos 3\beta+\cos 3\gamma=3\cos(\alpha+\beta+\gamma),$$
$$\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma).$$

If $a_1=\text{cis } 2\alpha_1$, $a_2=\text{cis } 2\alpha_2$, $a_3=\text{cis } 2\alpha_3$, $a_4=\text{cis } 2\alpha_4$, prove that

17. $(a_1+a_2)(a_3+a_4)=4\cos(\alpha_1-\alpha_2)\cos(\alpha_3-\alpha_4)\,\text{cis}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)$.

18. $a_1a_2+a_3a_4=2\cos(\alpha_1+\alpha_2-\alpha_3-\alpha_4)\,\text{cis}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)$.

19. Use the identity

$$(a_1-a_2)(a_3-a_4)+(a_2-a_3)(a_1-a_4)+(a_3-a_1)(a_2-a_4)=0,$$

to prove $\sin(\alpha_1-\alpha_2)\sin(\alpha_3-\alpha_4)+\ldots=0$.

20. Find all the values of

(i) $(\cos\alpha+i\sin\alpha)^{3/4}$ (ii) $(\cos\alpha+i\sin\alpha)^{6/8}$.

ia. Defining $(\text{cis }\alpha)^{3/4}$, as $[(\text{cis }\alpha)^3]^{1/4}$, we have

$$(\text{cis }\alpha)^{3/4}=(\text{cis }3\alpha)^{1/4}=[\text{cis }(3\alpha+2r\pi)]^{1/4}$$

$$=\text{cis}\,\frac{3\alpha+2r\pi}{4},\text{ where } r=0,\ 1,\ 2,\ 3;$$

i.e., $$\text{cis}\,\frac{3\alpha}{4},\ \text{cis}\,\frac{3\alpha+2\pi}{4},\ \text{cis}\,\frac{3\alpha+4\pi}{4},\ \text{cis}\,\frac{3\alpha+6\pi}{4}.$$

ib. If we define $(\text{cis }\alpha)^{3/4}$ as $[(\text{cis }\alpha)^{1/4}]^3$

$$[(\text{cis }\alpha)^{1/4}]^3=[\{\text{cis }(\alpha+2r\pi)\}^{1/4}]^3$$

$$=\left[\text{cis}\,\frac{\alpha+2r\pi}{4}\right]^3 \qquad r=0,\ 1,\ 2,\ 3$$

$$=\text{cis}\,\frac{3\alpha+6\pi r}{4}, \qquad r=0,\ 1,\ 2,\ 3$$

or $$\text{cis}\,\frac{3\alpha}{4},\ \text{cis}\,\frac{3\alpha+6\pi}{4},\ \text{cis}\,\frac{3\alpha+12\pi r}{4},\ \text{cis}\,\frac{3\alpha+18\pi r}{4},$$

i.e., $$\text{cis}\,\frac{3\alpha}{4},\ \text{cis}\,\frac{3\alpha+6\pi}{4},\ \text{cis}\,\frac{3\alpha+4\pi r}{4},\ \text{cis}\,\frac{3\alpha+2\pi r}{4};$$

and these are the same as these found in (ia).

iia. If the index 6/8 he reduced to its lowest term, the values of $(\text{cis }\alpha)^{6/8}$ are the same as those of $(\text{cis }\alpha)^{3/4}$.

iib. If the index 6/8 be taken as it is, the eight values of $(\text{cis }\alpha)^{6/8}$ according to the second definition, *viz.*

$$(\text{cis }\alpha)^{6/8}=[(\text{cis }\alpha)^{1/8}]^6=[\{\text{cis}(\alpha+2\pi r)\}^{1/8}]^6=\text{cis}\,\tfrac{6}{8}(\alpha+2\pi r)$$

are $\text{cis}\,\frac{6}{8}(\alpha)$ $\text{cis}\,\frac{6}{8}(\alpha+2\pi)$, $\text{cis}\,\frac{6}{8}(\alpha+4\pi)$, $\text{cis}\,\frac{6}{8}(\alpha+6\pi)$, $\text{cis}\,\frac{6}{8}(\alpha+8\pi)$, $\text{cis}\,\frac{6}{8}(\alpha+10\pi)$, $\text{cis}\,\frac{6}{8}(\alpha+12\pi)$, $\text{cis}\,\frac{6}{8}(\alpha+14\pi)$.

Hence we notice that the first four values are the same as those already found in (iia), and the last four are repetitions of the first four.

In the general case, if the index be p/q, where p and q have the greatest common factor k, so that $p=p'k$, $q=q'k$,

$$(\text{cis } a)^{p/q}=[\text{cis}(a+2\pi r)]^{p/q} \quad (\text{where } r=0,\ 1,\ 2\ldots\ q-1)$$

$$=\text{cis } \frac{p}{q}(a+2\pi r)$$

$$=\text{cis } \frac{p'}{q'}(a+2\pi r),\ r=0,\ 1,\ 2,\ \ldots q-1.$$

It is evident that these $q=kq'$ values are k repetitions of the q' values for $r=0.\ 1.\ 2,\ \ldots(q'-1)$,

so that this definition will give the same values of $(\text{cis } a)^{p/q}$, although each is repeated k times.

iic. Again, we may define

$$(\text{cis } a)^{6/8}=\sqrt[8]{[(\text{cis } a)^6]}=[(\text{cis } a)^6]^{1/8}=(\text{cis } 6a)^{1/8}$$

$$=[\text{cis } (6a+2\pi r)]^{1/8}$$

$$=\text{cis } \frac{6a+2\pi r}{8}.\quad (r=0,\ 1,\ 2,\ 3,\ 4,\ 5.\ 6,\ 7)$$

These values for $r=0,\ 2,\ 4;\ 6$ are the same as those already found in (i), or $(ii\ a,\ b)$. The values for $r=1,\ 3,\ 5,\ 7$, *viz.*

$$\text{cis } \frac{3a+\pi}{4},\ \text{cis } \frac{3a+3\pi}{4},\ \text{cis } \frac{3a+5\pi}{4},\ \text{cis } \frac{3a+7\pi}{4}$$

are not admissible.

For $$\left[\text{cis } \frac{3a+\pi r}{4}\right]^{8/6} \qquad r=1,\ 3,\ 5,\ 7$$

$$=\text{cis } \frac{3a+\pi r}{3} \quad =\text{cis } \left(a+\frac{\pi r}{3}\right)\neq\text{cis } a.$$

To explain these extraneous values, we notice that

$$a^{p/q}=a^{p'k/q'k}=(a^{p'/q'})^{k/k}$$

$$=[(a^{p'/q'})^k]^{\frac{1}{k}}$$

and each of the $q'=q/k$ value of $a^{p'/q'}$ will give rise to k different values. Thus we have $q'=q/k$ sets of k values, and only one value in each set will be admissible. Rejecting such values, we have the same set of values as those of $(\text{cis } a)^{p'/q'}$.

It is therefore, advisable to reduce p/q to its lowest terms before we proceed to find the value of $(\text{cis } a)^{p/q}$.

21. Find all the values of $(\cos a+i \sin a)^{2/3}$.

22. Find the continued product of the five values of

$$(\cos a+i \sin a)^{3/5}.$$

23. Show that $\{(\cos\theta-\cos\phi)+i(\sin\theta-\sin\phi)\}^n$
$+\{(\cos\theta-\cos\phi)-i(\sin\theta-\sin\phi)\}^n$
$=2^{n+1}\sin^n\frac{\theta-\phi}{2}\cos^n\frac{\pi+\theta+\phi}{2}$.

24. Prove that

$$(\sin x+i\cos x)^n=\cos n\left(\frac{\pi}{2}-x\right)+i\sin n\left(\frac{\pi}{2}-n\right),$$

and $(1+\sin\phi+i\cos\phi)^n(1+\sin\phi-i\cos\phi)^{-n}$

$$=\cos\left(\frac{n\pi}{2}-n\phi\right)+i\sin\left(\frac{n\pi}{2}-n\phi\right).$$

25. If $2\cos\theta=x+\frac{1}{x}$ and $2\cos\phi=y+\frac{1}{y}$,

prove that one of the values of $x^my^n+\frac{1}{x^my^n}$ is

$$2\cos(m\theta+n\phi).$$

26. Prove that

$$(a+ib)^{m/n}+(a-ib)^{m/n}=2(a^2+b^2)^{m/2n}\cos\left(\frac{m}{n}\tan^{-1}\frac{b}{a}\right).$$

27. Prove that $\left(\frac{1+i\sqrt{3}}{2}\right)^n+\left(\frac{-1-i\sqrt{3}}{2}\right)^n$

has the value -1, if $n=3k\pm1$; and the value 2, if $n=3k$, where k is an integer.

10·3. Sum Formulae for Circular Functions.

We have

$\cos(\theta_1+\theta_2+\cdots+\theta_n)+i\sin(\theta_1+\theta_2+\cdots+\theta_n)$
$=\text{cis}\,\theta_1.\ \text{cis}\,\theta_2\cdots\text{cis}\,\theta_n$
$=\cos\theta_1\cos\theta_2\cdots\cos\theta_n(1+i\tan\theta_1)(1+i\tan\theta_2)\cdots(1+i\tan\theta_n)$
$=\cos\theta_1\cos\theta_2\ldots\cos\theta_n[1+it_1-t_2-it_3+t_4+\cdots]$,
where $t_1=\tan\theta_1+\tan\theta_2+\cdots+\tan\theta_n$,
$t_2=\Sigma\tan\theta_1\tan\theta_2$,
$t_3=\Sigma\tan\theta_1\tan\theta_2\tan\theta_3$, etc.

Separating the real and imaginary parts,

$$\cos(\theta_1+\theta_2+\cdots+\theta_n)=\cos\theta_1\cos\theta_2\cdots+\cos\theta_n(1-t_2+t_4-t_6+\cdots), \quad (1)$$

$$\sin(\theta_1+\theta_2\cdots+\theta_n)=\cos\theta_1\cos\theta_2\cdots\cos\theta_n(t_1-t_3+t_5-t_7+\cdots). \quad (2)$$

Division gives

$$\tan(\theta_1+\theta_2+\cdots+\theta_n)=\frac{t_1-t_3+t_5-t_7+\cdots}{1-t_2+t_4-t_6+\cdots} \quad (3)$$

note $n C_2 = \frac{L_n}{\lfloor 2 \lfloor n-2}$

10·4. Expression of cos $n\theta$, sin $n\theta$ in terms of cos θ and sin θ, n being a positive integer.

Denote sin θ, cos θ, tan θ by s, c, t respectively.

Case I. When n is even and $=2m$.

$$\text{cis } 2m\theta=(\text{cis }\theta)^{2m}=(c+i\,s)^{2m}$$
$$=c^{2m}+{}^{2m}C_1\,c^{2m-1}\,(i\,s)+{}^{2m}C_2\,c^{2m-2}\,(i\,s)^2+\dots$$
$$+{}^{2m}C_{2m-1}\,c\,(i\,s)^{2m-1}+{}^{2m}C_{2m}\,(i\,s)^{2m}.$$

Equating the real and imaginary parts,

$$\cos 2m\theta=c^{2m}-{}^{2m}C_2\,c^{2m-2}\,s^2+{}^{2m}C_4\,c^{2m-4}\,s^4+\dots$$
$$+(-1)^m\,{}^{2m}C_{2m}s^{2m}, \quad (4)$$
$$\sin 2m\theta={}^{2m}C_1\,c^{2m-1}\,s-{}^{2m}C_3c^{2m-3}\,s^3+\dots$$
$$+(-1)^{m-1}\,{}^{2m}C_{2m-1}\,c\,s^{2m-1}. \quad (5)$$
$$\therefore \quad \tan 2m\theta=\frac{{}^{2m}C_1\,t-{}^{2m}C_3\,t^3+\dots+(-1)^{m-1}\,{}^{2m}C_{2m-1}\,t^{2m-1}}{1-{}^{2m}C_2\,t^2+{}^{2m}C_4\,t^4\dots+(-1)^m\,{}^{2m}C_{2m}\,t^{2m}}. \quad (6)$$

Case II. Similarly, when n is odd and $=2m+1$.

$$\text{Cis } (2m+1)\theta=(c+i\,s)^{2m+1}$$
$$=c^{2m+1}+{}^{2m+1}C_1\,c^{2m}\,(i\,s)+{}^{2m+1}C_2\,c^{2m-1}\,(i\,s)^2+\dots$$
$$+{}^{2m+1}C_{2m+1}\,(i\,s)^{2m+1},$$

Hence

$$\cos (2m+1)\theta=c^{2m+1}-{}^{2m+1}C_2\,c^{2m-1}\,s^2+\dots$$
$$+{}^{2m+1}C_{2m}(-1)^m\,c\,s^{2m}, \quad (7)$$
$$\sin (2m+1)\theta={}^{2m+1}C_1\,c^{2m}s-{}^{2m+1}C_3\,c^{2m-2}s^3+\dots$$
$$+{}^{2m+1}C_{2m+1}\,(-1)^m\,s^{2m+1}. \quad (8)$$

By division we obtain

$$\tan (2m+1)\theta=\frac{{}^{2m+1}C_1\,t-{}^{2m+1}C_3\,t^3+{}^{2m+1}C_5\,t^5-\dots+(-1)^m\,t^{2m+1}}{1-{}^{2m+1}C_2\,t^2+{}^{2m+1}C_4\,t^4-\dots+(-1)^m\,{}^{2m+1}C_{2m}\,t^{2m}} \quad (9)$$

10·41. Expressions for cos $n\theta$ and sin $n\theta$ in series of descending powers of cos θ or sin θ.

I. When n is *an even positive integer*, expressions for cos n θ [10·4 ; (4), (7)] involve only even powers of s, and replacing s^2 by $1-c^2$, it follows that cos $n\theta$ can always be expressed in descending powers of cos θ. Each power of cos θ is less by two than the preceding power.

II. If n is *odd*, the expression for cos $n\theta$/cos θ in [10·4, (7)] contains even powers of cos θ, and, therefore, cos nθ/cos θ can be expressed in a finite series of descending powers of sin θ. Each power is less than the preceding power by two, as in Case I. This will hold in all subsequent cases also.

III. Again if n is *even*, $\sin n\theta/\sin\theta$, and $\sin n\theta/\cos\theta$ can be expressed in a finite series of descending powers of $\cos\theta$ and $\sin\theta$ respectively.

IV. If n is odd, $\sin n\theta$ can be expressed in a finite series of $\sin\theta$.

The general expressions for these series in ascending powers of $\cos\theta$ and $\sin\theta$ will be obtained in the next chapter. Here we have only pointed out the possibility of such expansions and the set of exercises that follows illustrates the application of the above principles.

Examples X (*c*)

1. Express $\cos 5\theta$, $\sin 5\theta$ and $\tan 5\theta$ in terms of $\cos\theta$, $\sin\theta$ and $\tan\theta$.

Solution. Let c stand for $\cos\theta$ and s for $\sin\theta$

$$\cos 5\theta+i\sin 5\theta=(c+is)^5$$
$$=c^5+5c_1\,c^4(is)+{}^5C_2\,c^3(is)^2+{}^5C_3\,c^2(is)^3+{}^5C_4\,c(is)^4+{}^5C_5(is)^5$$
$$=c^5+5i\,c^4s-10c^3s^2-10i\,c^2s^3+5cs^4+is^5$$
$$\therefore\ \cos 5\theta=c^5-10c^3s^3+5cs^4=c^5-10c^3(1-c^2)+5c(1-c^2)^2$$
$$=16c^5-20c^3+5c,$$
$$\sin 5\theta=5c^4s-10c^2s^3+s^5=5(1-2s^2+s^4)s-10(1-s^2)s^3+s^5$$
$$=5s-20s^3+16s^5.$$

Dividing $\sin 5\theta$ by $\cos 5\theta$,

$$\tan 5\theta=\frac{5\,c^4s-10c^2\,s^3+s^5}{c^5-10c^3s^2+5cs^4}$$
$$=\frac{5t-10t^3+t^5}{1-10\,t^2+5\,t^4},\quad\text{where } t=\frac{s}{c}=\tan\theta.$$

2. Write down the value of $\tan 7\theta$ in terms of $\tan\theta$. [*P. U. 1937*]

If c and s stand for $\cos\theta$ and $\sin\theta$ respectively, prove the following.

3. $\sin 6\theta=6c^5s-20c^3s^3+6cs^5$.
4. $\cos 8\theta=c^8-28c^6s^2+70c^4s^4-28c^2s^6+s^8$.
5. $\cos 9\theta\sec\theta=c^8-36c^6s^2+126\,c^4s^4-84\,c^2s^6+9s^8$.
6. $\operatorname{cosec}\theta\sin 7\theta=7c^6-35\,c^4s^2+21\,c^2s^4-s^6$.

Prove the following, where $c=\cos\theta$, $s=\sin\theta$.

7. $\cos 7\theta=64\,c^7-112\,c^5+56\,c^3-7c$.
8. $\sin 7\theta=7s-56\,s^3+112\,s^5-64\,s^7$.
9. $\sin 8\theta.\operatorname{cosec}\theta=128c^7-192\,c^5+80c^3-8c$.
10. $\cos 8\theta=128c^8-256\,c^6+160\,c^4-32c^2+1$.
11. When n is even, prove that the last terms in the expansions of $\sin n\theta$ and $\cos n\theta$ are respectively

$$n(-1)^{\frac{n-2}{2}}\,c\,s^{n-1}\text{ and }(-1)^{\frac{n}{2}}s^n,$$

where $c\equiv\cos\theta$, $s\equiv\sin\theta$.

12. If n is odd, prove that the last terms in the expansions of $\sin n\theta$, $\cos n\theta$ respectively, are, with the usual notation,

$$(-1)^{\frac{n-1}{2}} s^n \quad \text{and} \quad n(-1)^{\frac{n-1}{2}} c\, s^{n-1},$$

13. Prove the following (n being a positive integer)

$$1-\frac{n(n-1)}{2!}+\frac{n(n-1)(n-2)(n-3)}{4!}-\ldots\ldots=2^{\frac{n}{2}}\cos\frac{n\pi}{4},$$

$$n-\frac{n(n-1)(n-2)}{3!}+\frac{n(n-1)(n-2)(n-3)(n-4)}{5!}$$

$$-\ldots\ldots=2^{\frac{n}{2}}\sin\frac{n\pi}{4}.$$

14. If n is a positive odd integer, the sum of the products taken two together of the $(n-1)$ quantities

$$\tan\frac{\pi}{n}, \tan\frac{2\pi}{n}, \ldots\ldots\tan\frac{n-1}{n}\pi$$

is $\dfrac{n(1-n)}{2}$.

15. Show that

$$\tan\frac{\theta}{n}+\tan\frac{\theta+\pi}{n}+\tan\frac{\theta+2\pi}{n}+\ldots\ldots+\tan\frac{\theta+n-1}{n}\pi.$$

$$=-n\cot\theta \text{ or } n\tan\theta,$$

according as n is even or odd.

16. Prove that $\cos\frac{2\pi}{7}$, $\cos\frac{4\pi}{7}$, $\cos\frac{8\pi}{7}$ are the roots of the equation $8x^3+4x^2-4x-1=0$.

First Solution. If θ denotes any one of these angles, then

$$7\theta=2\pi \text{ or } 4\pi \text{ or } 8\pi.$$

$\therefore$ $\sin 4\theta=-\sin 3\theta$

or $4\sin\theta\cos\theta(2\cos^2\theta-1)\equiv-(3\sin\theta-4\sin^3\theta)$

$\therefore$ $4\cos\theta(2\cos^2\theta-1)=-(3-4\sin^2\theta)$ [$\because$ $\sin\theta\neq0$]

$$=-3+4(1-\cos^2\theta).$$

Hence $\cos\theta$ is a root of

$$8x^3+4x^2-4x-1=0.$$

Second Solution. Let θ be any one of the angles

$$\frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}, \frac{8\pi}{7}, \frac{10\pi}{7}, \frac{12\pi}{7}.$$

If

$$y=\cos\theta+i\sin\theta,$$

$$y^7=\cos 7\theta+i\sin 7\theta=1$$

then

$$y^7-1=(y-1)(y^6+y^5+y^4+y^3+y^2+y+1)=0.$$

Now $y=1$ corresponds to $\theta=2k\pi$ (k being an integer)

$\therefore$ the roots of the equation

$$y^6+y^5+y^4+y^3+y^2+y+1=0 \qquad (1)$$

are $\cos\theta+i\sin\theta$, θ being any one of the above six angles.

The above equation is reciprocal ; its roots occur therefore in reciprocal pairs as

$$\text{cis}\ \frac{2\pi}{7},\ \text{cis}\ \frac{12\pi}{7};\ \text{cis}\ \frac{4\pi}{7},\ \text{cis}\ \frac{10\pi}{7};\ \text{cis}\ \frac{8\pi}{7};\ \text{cis}\ \frac{6\pi}{7}.$$

i.e., $\text{cis}\ \frac{2\pi}{7},\ \text{cis}\ \frac{-2\pi}{7};\ \text{cis}\ \frac{4\pi}{7},\ \text{cis}\ \frac{-4\pi}{7};\ \text{cis}\ \frac{8\pi}{7},\ \text{cis}\ \frac{-8\pi}{7}.$

Put $$2x=y+\frac{1}{y}=\text{cis}\ \theta+\frac{1}{\text{cis}\ \theta}=\text{cis}\ \theta+\text{cis}\ (-\theta)$$

$$=2\cos\theta,$$

where $$\theta=\frac{2\pi}{7},\ \frac{4\pi}{7}\ \text{or}\ \frac{8\pi}{7}.$$

Since $$2x=y+\frac{1}{y},$$

$$y^2+\frac{1}{y^2}=4x^2-2,\quad y^3+\frac{1}{y^3}=8x^3-6x,$$

Writing the equation (1) as

$$y^3+\frac{1}{y^3}+y^2+\frac{1}{y^2}+y+\frac{1}{y}+1=0,$$

which transforms to

$$(8x^3-6x)+(4x^2-2)+2x=0$$

or $$8x^3+4x^2-4x-1=0,$$

the roots of which are $\cos\theta$, where θ has any of the values

$$\frac{2\pi}{7},\ \frac{4\pi}{7},\ \frac{8\pi}{7}.$$

We incidentally prove that

$$\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}=-\frac{1}{2},$$

$$\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}+\cos\frac{4\pi}{7}\cos\frac{8\pi}{7}+\cos\frac{8\pi}{7}.\cos\frac{2\pi}{7}=\frac{1}{2},$$

$$\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{8\pi}{7}=-\frac{1}{8}.$$

17. Prove that the roots of the equation

$$8x^3-4x^2-4x+1=0$$

are $$\cos\frac{\pi}{7},\ \cos\frac{3\pi}{7},\ \cos\frac{5\pi}{7};$$

and deduce $$\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}=\frac{1}{2},$$

$$\cos\frac{\pi}{7}\cos\frac{3\pi}{7}+\cos\frac{3\pi}{7}\cos\frac{5\pi}{7}+\cos\frac{5\pi}{7}\cos\frac{\pi}{7}=-\frac{1}{2}.$$

18. Form the equation whose roots are

$$\sec^2\frac{2\pi}{7},\ \sec^2\frac{4\pi}{7},\ \sec^2\frac{8\pi}{7}.$$

Solution. From Ex. 16, the roots of the equation

$$8x^3+4x^2-4x-1=0 \qquad (1)$$

are $$\cos\frac{2\pi}{7},\ \cos\frac{4\pi}{7},\ \cos\frac{8\pi}{7}.$$

If a root of this equation be denoted by x and the root of the proposed equation by z, the transformation which connects the roots of the two equations is

$$z=\frac{1}{x^2} \quad \text{or} \quad x^2=\frac{1}{z}. \qquad (2)$$

x can now be eliminated between (1) and (2).

Thus $$x\left(\frac{8}{z}-4\right)=1-\frac{4}{z}.$$

$$\therefore\quad x^2\left(\frac{64}{z^2}-\frac{64}{z}+16\right)=1-\frac{8}{z}+\frac{16}{z^2}$$

or $$\frac{1}{z}\left(\frac{64}{z^2}-\frac{64}{z}+16\right)=1-\frac{8}{z}+\frac{16}{z^2},$$

whence $$16z^2-64z+64=z^3-8z^2+16z,$$

or $$z^3-24z^2+80z-64=0.$$

19. Show that the roots of the equation

$$x^3-24x^2+80x-64=0$$

are $$\sec^2\frac{\pi}{7},\ \sec^2\frac{3\pi}{7},\ \sec^2\frac{5\pi}{7}.$$

20. Show that the equation whose roots are

$$\tan^2\frac{2\pi}{7},\ \tan^2\frac{4\pi}{7},\ \tan^2\frac{8\pi}{7} \text{ is } x^3-21x^2+35x-7=0.$$

[**Hint.** Transform the equation in z of Ex. 18 by the transformation $z=1+x$, or directly from 10·4, the result may be obtained as follows :—

If θ is any one of the angles,

$$\tan 7\theta=\frac{7t-{}^7C_3t^2+{}^7C_5t^5-{}^7C_7t^7}{1-{}^7C_2t^2+{}^7C_4t^4-{}^7C_6t^6}=0, \qquad [t=\tan\theta]$$

$$\therefore\quad 7-35t^2+21t^4-t^6=0,$$

after removing t which is different from zero.

Putting $t^2=x$, $\quad x^3-21x^2+35x-7=0$
is the required equation.]

21. Find the equation whose roots are

$$\tan\frac{\pi}{9},\ \tan\frac{4\pi}{9},\ \tan\frac{7\pi}{9}.$$

[Solution. If θ denotes any one of these angles,

$$\tan 3\theta=\sqrt{3}.$$

$$\therefore\quad \frac{3t-t^3}{1-3t^2}=\sqrt{3}, \qquad [t=\tan\theta.$$

Hence the required equation is $t^3-3\sqrt{3}t^2-t+\sqrt{3}=0$].

Examples X (*d*)

1. Find all the values of $(3+4i)^{\frac{1}{4}}$. Hence solve the simultaneous equations $x^4-6x^2y^2+y^4=3$, $xy(x^2-y^2)=4$, for real values of x and y.

2. If n is a positive integer and $x_n+iy_n=(1+i\sqrt{3})^n$, show that
$$x_{n-1}y_n-x_ny_{n-1}=4^{n-1}\sqrt{3}.$$

3. Find the continued product of the four values of

$$\left(\cos\frac{\pi}{3}+i\ \sin\frac{\pi}{3}\right)^{3/4}.$$

4. Find the continued product of the q values of

$$\left(\cos\frac{\pi}{p}+i\ \sin\frac{\pi}{p}\right)^{p/q}.$$

5. By means of the equation $\tan 11x=0$, or otherwise, prove that

$$\tan^2\frac{\pi}{11}+\tan^2\frac{2\pi}{11}+\tan^2\frac{3\pi}{11}+\tan^2\frac{4\pi}{11}+\tan^2\frac{5\pi}{11}=55,$$

$$\tan\frac{\pi}{11}\tan\frac{2\pi}{11}\tan\frac{3\pi}{11}\tan\frac{4\pi}{11}\cdot\frac{5\pi}{11}=\sqrt{11}.$$

6. Prove that $2\cos\frac{2\pi}{5}$, $2\cos\frac{4\pi}{5}$, $2\cos\frac{6\pi}{5}$, $2\cos\frac{8\pi}{5}$ are the roots of the equation $x^4+2x^3-x^2-2x+1=0$. [*P. U. 1937*]

7. Prove that $\sin\frac{\pi}{14}$ is a root of the equation
$$64x^6-80x^4+24x^2-1=0.$$

8. If $x_r=\cos\frac{\pi}{2^r}+i\ \sin\frac{\pi}{2^r}$, prove that $x_1x_2\ldots$ad inf.$=\cos\pi$.

9. Prove that the roots of the equation $x^3-\frac{\sqrt{7}}{2}x^2+\frac{\sqrt{7}}{8}=0$ are $\sin\frac{2\pi}{7}$, $\sin\frac{4\pi}{7}$ and $\sin\frac{8\pi}{7}$.

Prove the following

10. $\cos\frac{\pi}{11}+\cos\frac{3\pi}{11}+\cos\frac{5\pi}{11}+\cos\frac{7\pi}{11}+\cos\frac{9\pi}{11}=\frac{1}{2}$.

11. $\cot^2\frac{\pi}{11}+\cot^2\frac{2\pi}{11}+\cot^2\frac{3\pi}{11}+\cot^2\frac{4\pi}{11}+\cot^2\frac{5\pi}{11}=15$,

12. $\cos\frac{2\pi}{13}+\cos\frac{6\pi}{13}+\cos\frac{18\pi}{13}=\frac{\sqrt{3}-1}{4}$,

13. $\cos\frac{10\pi}{13}+\cos\frac{14\pi}{13}+\cos\frac{22\pi}{13}=\frac{-\sqrt{3}-i}{4}$.

14. $\cos\frac{\pi}{15}+\cos\frac{7\pi}{15}+\cos\frac{11\pi}{15}+\cos\frac{13\pi}{15}=-\frac{1}{2}$.

15. $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}=(\frac{1}{2})^7$.

16. Obtain an expression for $\cos 5\theta$ as a polynomial in $\cos\theta$.

Show that $\cos\frac{\pi}{10}=\frac{1}{2}\sqrt{\left(\frac{5+\sqrt{5}}{2}\right)}$, $\cos\frac{3\pi}{10}=\frac{1}{2}\sqrt{\left(\frac{5-\sqrt{5}}{2}\right)}$.

17. Find the continued product of the six values of

$$\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)^{\frac{5}{6}}.$$

18. Prove that $x=2\sin 10°$ is a root of the equation $x^3-3x+1=0$, and find the other two roots.

CHAPTER XI

FINITE EXPANSIONS AND PRODUCTS

11·1 To express $\cos^n\theta$ in ter ıs of multiples of θ, n being a positive integer.

Let $$x=\cos\theta+i\sin\theta,$$

then $$\frac{1}{x}=\cos\theta-i\sin\theta,$$

and $$x^n=(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta,$$

$$x^{-n}=(\cos\theta-i\sin\theta)^n=\cos n\theta-i\sin\theta.$$

$$\therefore\ x+\frac{1}{x}=2\cos\theta,\quad x-\frac{1}{x}=2i\sin\theta,\ x^n+\frac{1}{x^n}=2\cos n\theta,$$

$$x^n-\frac{1}{x^n}=2i\sin n\theta.$$

Hence $$(2\cos\theta)^n=\left(x+\frac{1}{x}\right)^n$$

$$=x^n+{}^nC_1x^{n-1}\frac{1}{x}+{}^nC_2x^{n-2}\frac{1}{x^2}+\cdots+{}^nC_{n-1}\ x\frac{1}{x^{n-1}}+{}^nC_n\frac{1}{x^n}$$

$$=\left(x^n+\frac{1}{x^n}\right)+{}^nC_1\left(x^{n-2}+\frac{1}{x^{n-2}}\right)+{}^nC_2\left(x^{n-4}+\frac{1}{x^{n-4}}\right)+\cdots$$

$$\therefore\ 2^{n-1}\cos^n\theta=\cos n\theta+n\cos(n-2)\theta+\frac{n(n-1)}{2!}\cos(n-4)\theta+\ldots(1)$$

If n is odd, the last term will be ${}^nC_{(n-1)/2}\cos\theta$; and $\frac{1}{2}\,{}^nC_{n/2}$, if n is even.

11·11. To express $\sin^n\theta$ in terms of sines or cosines of multiples of θ, according as n is an even or odd positive integer.

(i) $$(2i\sin\theta)^n=\left(x-\frac{1}{x}\right)^n$$

$$x^n-{}^nC_1\,x^{n-1}\frac{1}{x}+{}^nC_2\,x^{n-2}\cdot\frac{1}{x^2}+\cdots+{}^nC_2\,x^2\cdot\frac{1}{x^{n-2}}$$

$$-{}^nC_1\,x\cdot\frac{1}{x^{n-1}}+{}^nC_n\,\frac{1}{x^n}.$$

When n is *even*,

$$(-1)^{\frac{n}{2}}\,2^n\sin^n\theta=\left(x^n+\frac{1}{x^n}\right)-{}^nC_1\left(x^{n-2}+\frac{1}{x^{n-2}}\right)+{}^nC_4\left(x^{n-4}+\frac{1}{x^{n-4}}\right)+\cdots$$

i.e., $(-1)^{\frac{n}{2}}2^{n-1}\sin^n\theta$

$$=\cos n\theta-n\cos(n-2)\theta+\frac{n(n-1)}{2!}\cos(n-4)\theta\ldots$$

the last term being $(-1)^{\frac{n}{2}}\,{}^nC_{n/2}$. (2)

(ii) If n is *odd*,

$$(2i\sin\theta)^n=x^n-{}^nC_1\,x^{n-1}\cdot\frac{1}{x}+{}^nC_2\,x^{n-2}\cdot\frac{1}{x^2}+\cdots+{}^nC_1x\cdot\frac{1}{x^{n-1}}-{}^nC_n\frac{1}{x^n}$$

$$=\left(x^n-\frac{1}{x^n}\right)-{}^nC_1\left(x^{n-2}-\frac{1}{x^{n-2}}\right)+{}^nC_2\left(x^{n-4}-\frac{1}{x^{n-4}}\right)-\cdots$$

$\therefore\ 2^{n-1}(-1)^{\frac{n-1}{2}}\sin{}^n\theta$

$$=\sin n\theta-n\sin(n-2)\theta+\frac{n(n-1)}{2}\sin(n-4)\theta-\cdots,$$

the last term being $(-1)^{\frac{n-1}{2}}\,{}^nC_{\frac{n-1}{2}}\sin\theta$. (3)

EXAMPLES XI (*a*)

1. Expand $\cos^6\theta$ in a series of cosines of multiples of θ.

Solution. Let $x=\cos\theta+i\sin\theta$, then $x^{-1}=\cos\theta-i\sin\theta$ and $x^n=\cos n\theta+i\sin n\theta$, $x^{-n}=\cos n\theta-i\sin n\theta$

$$\therefore\ (2\cos\theta)^6=\left(x+\frac{1}{x}\right)^6=x^6+{}^6C_1x^5.\frac{1}{x}+{}^6C_2x^4.\frac{1}{x^2}+{}^6C_3x^3.\frac{1}{x^3}+{}^6C_4x^2.\frac{1}{x^4}+{}^6C_5x\cdot\frac{1}{x^5}+{}^6C_6\cdot\frac{1}{x^6}$$

$$=\left(x^6+\frac{1}{x^6}\right)+6\left(x^4+\frac{1}{x^4}\right)+15\left(x^2+\frac{1}{x^2}\right)+20.$$

$$=2\cos6\theta+6(2\cos4\theta)+15(2\cos2\theta)+20$$

$$\therefore\ \cos^6\theta=\frac{1}{2^5}\{\cos6\theta+6\cos4\theta+15\cos2\theta+10\}.$$

Prove the following :—

2. $\cos^7\theta=\frac{1}{64}\{\cos 7\theta+7\cos 5\theta+21\cos 3\theta+35\cos\theta\}$.

3. $\sin^7\theta=-\frac{1}{64}\{\sin 7\theta-7\sin 5\theta+21\sin 3\theta-35\sin\theta\}$.

4. $2^6\cos^3\theta\sin^4\theta=\cos 7\theta-\cos 5\theta-3\cos 3\theta+3\cos\theta$.

Solution. $(2\cos\theta)^3(2\sin\theta)^4=\left(x+\frac{1}{x}\right)^3\left(x-\frac{1}{x}\right)^4$ [as in (1)

$$=\left(x^3+3x+\frac{3}{x}+\frac{1}{x^3}\right)\left(x^4-4x^2+6-\frac{4}{x^2}+\frac{1}{x^4}\right)$$

$$=\left(x^7+\frac{1}{x^7}\right)-\left(x^5+\frac{1}{x^5}\right)-3\left(x^3+\frac{1}{x^3}\right)+3\left(x+\frac{1}{x}\right).$$

$\therefore$ $2^6\cos^3\theta\sin^4\theta=\cos 7\theta-\cos 5\theta-3\cos 3\theta+3\cos\theta$.

5. $32\sin^4\theta\cos^2\theta=\cos 6\theta-2\cos 4\theta-\cos 2\theta+2$.

6. $2^{11}\sin^5\theta\cos^7\theta=\sin 12\theta+2\sin 10\theta-4\sin 8\theta-10\sin 6\theta+5\sin 4\theta+20\sin 2\theta$.

7. $2^9\cos^7\theta\sin^3\theta=-\sin 10\theta-4\sin 8\theta-3\sin 6\theta+8\sin 4\theta+14\sin 2\theta$.

11·2. Factors of cos nθ.

In Art. 10·4. it has been proved that

$$\cos n\theta=c^n-{}^nC_2\,c^{n-2}\,s^2+{}^nC_4\,c^{n-4}\,s^4+\ldots\ldots \qquad (1)$$

Thus $\cos n\theta$ is a polynomial in $\cos\theta$ of degree n.

Also the coefficient of $\cos^n\theta$ is

$$1+{}^nC_2+{}^nC_4+\ldots\ldots=\tfrac{1}{2}[(1+1)^n+(1-1)^n]=2^{n-1}.$$

Hence $\cos n\theta$ can be expressed as the product of n linear factors. If $a_1, a_2\ldots a_n$ be the roots of the equation corresponding to (1),

$$\cos n\theta=2^{n-1}(\cos\theta-a_1)(\cos\theta-a_2)\ldots\ldots(\cos\theta-a_n).$$

Now $\cos n\theta=0$, when $\theta=\frac{(2r+1)\pi}{2n}$, $(r=0, 1, 2,\ldots\ldots n-1)$, and all these are different angles. Hence

$$\cos n\theta=2^{n-1}\left(\cos\theta-\cos\frac{\pi}{2n}\right)\left(\cos\theta-\cos\frac{3\pi}{2n}\right)\ldots\ldots$$
$$\times\left(\cos\theta-\cos\frac{2n-3}{2n}\pi\right)\left(\cos\theta-\cos\frac{2n-1}{2n}\pi\right).$$

Since $\cos\frac{2n-1}{2n}\pi=-\cos\frac{\pi}{2n}$, $\cos\frac{2n-3}{2n}\pi=-\cos\frac{3\pi}{2n}\ldots\ldots$,

we may arrange these factors in pairs, and thus obtain

$$\cos n\theta=2^{n-1}\left(\cos^2\theta-\cos^2\frac{\pi}{2n}\right)\left(\cos^2\theta-\cos^2\frac{3\pi}{2n}\right)\ldots\left(\cos^2\theta-\cos^2\frac{n-1}{2n}\pi\right) \qquad (2)$$

when n is even ... ...

$$\cos n\theta = 2^{n-1}\left(\cos^2\theta - \cos^2\frac{\pi}{2n}\right)\left(\cos^2\theta - \cos^2\frac{3\pi}{2n}\right)\ldots\ldots$$

$$\left(\cos^2\theta - \cos^2\frac{n-2}{2n}\pi\right)\cos\theta, \text{ when } n \text{ is odd.} \quad \ldots \quad (3)$$

These expressions may be written as

$$\cos n\theta = 2^{n-1}\left(\sin^2\frac{\pi}{2n} - \sin^2\theta\right)\left(\sin^2\frac{3\pi}{2n} - \sin^2\theta\right)\ldots\left(\sin^2\frac{n-1}{2n}\pi - \sin^2\theta\right),$$

when n is even, and $\quad \ldots \quad (4)$.

$$\frac{\cos n\theta}{\cos\theta} = 2^{n-1}\left(\sin^2\frac{\pi}{2n} - \sin^2\theta\right)\left(\sin^2\frac{3\pi}{2n} - \sin^2\theta\right)\ldots\left(\sin^2\frac{n-2}{2n}\pi - \sin^2\theta\right)$$

when n is odd. $\quad \ldots \quad (5)$

Now let $\theta \to 0$.

When n is even, $1 = 2^{n-1}\sin^2\frac{\pi}{2n}\sin^2\frac{3\pi}{2n}\ldots\ldots\sin^2\frac{n-1}{2n}\pi, \ldots \quad (6)$

and when n is odd, $1 = 2^{n-1}\sin^2\frac{\pi}{2n}\sin^2\frac{3\pi}{2n}\ldots\ldots\sin^2\frac{n-2}{2n}\pi \ldots \quad (7)$

Dividing (4) by (6) and (5) by (7)

$$\cos n\theta = \left(1 - \frac{\sin^2\theta}{\sin^2\frac{\pi}{2n}}\right)\left(1 - \frac{\sin^2\theta}{\sin^2\frac{3\pi}{2n}}\right)\ldots\ldots\left(1 - \frac{\sin^2\theta}{\sin^2\frac{n-1}{2n}\pi}\right)\ldots \quad (8)$$

when n is even, and

$$\frac{\cos n\theta}{\cos\theta} = \left(1 - \frac{\sin^2\theta}{\sin^2\frac{\pi}{2n}}\right)\left(1 - \frac{\sin^2\theta}{\sin^2\frac{3\pi}{2n}}\right)\ldots\ldots\left(1 - \frac{\sin^2\theta}{\sin^2\frac{n-2}{2n}\pi}\right)\ldots \quad (9)$$

when n is odd.

Cor. Extracting the square root in (6) and (7)

$$1 = 2^{\frac{n-1}{2}}\sin\frac{\pi}{2n}\sin\frac{3\pi}{2n}\ldots\ldots\sin\frac{n-1}{2n}\pi, \quad \ldots \quad (10)$$

when n is even, and

$$1 = 2^{\frac{n-1}{2}}\sin\frac{\pi}{2n}\sin\frac{3\pi}{2n}\ldots\ldots\sin\frac{n-2}{2n}\pi \quad \ldots \quad (11)$$

when n is odd.

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In extracting the square root, the positive sign is taken, since all the angles $\frac{\pi}{2n}$, $\frac{3\pi}{2n}$.....arc acute and positive.

11·21. Factors of Sin nθ. In art. 10·4, it was shown that $$\frac{\sin n\theta}{\sin\theta} = {}^nC_1\, c^{n-1} - {}^nC_3\, c^{n-3}\, s^2 + \ldots.$$

As the expression on the right contains even powers of s, s^2 can be replaced by $1-c^2$, and so on ; so that $\sin n\theta/\sin\theta$ is a polynomial of degree $n-1$ in $\cos\theta$, n being a positive integer. Also the coefficient of $\cos^{n-1}\theta$ is

$${}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots\ldots = 2^{n-1}$$

Hence $\sin n\theta/\sin\theta$ can be expressed as the product of $n-1$ linear factors in $\cos\theta$ besides the numerical factor 2^{n-1}.

Now $\sin n\theta = 0$, when

$$\theta = \frac{r\pi}{n}, \qquad r = 0,\ 1,\ 2, \ldots\ldots n-1$$

$\therefore$ $\sin n\theta = 0$ when

$$\cos\theta = \cos\frac{r\pi}{n} \qquad r = 1,\ 2\ldots\ldots n-1,$$

the case $\theta = 0$ has been left out, because that corresponds to $\sin\theta$ being a factor of $\sin n\theta$.

Also all these values of $\cos\theta$ are different and no new value can be obtained by giving to r any other value. Hence

$$\frac{\sin n\theta}{\sin\theta} = 2^{n-1}\left(\cos\theta - \cos\frac{\pi}{n}\right)\left(\cos\theta - \cos\frac{2\pi}{n}\right)\ldots\ldots \times\left(\cos\theta - \cos\frac{n-1}{n}\pi\right).$$

Since $\cos\frac{n-p}{n}\pi = -\cos\frac{p\pi}{n}$, combining the factors equidistant from beginning and end, we obtain, when n is odd

$$\frac{\sin n\theta}{\sin\theta} = 2^{n-1}\left(\cos^2\theta - \cos^2\frac{\pi}{n}\right)\left(\cos^2\theta - \cos^2\frac{2\pi}{n}\right)\ldots\ldots \times\left(\cos^2\theta - \cos^2\frac{n-1}{2}\cdot\frac{\pi}{n}\right)$$

$$= 2^{n-1}\left(\sin^2\frac{\pi}{n} - \sin^2\theta\right)\left(\sin^2\frac{2\pi}{n} - \sin^2\theta\right)\ldots\ldots \times\left(\sin^2\frac{n-1}{2}\frac{\pi}{n} - \sin^2\theta\right). \quad \ldots \quad (12)$$

When n is even,

$$\frac{\sin n\theta}{\sin\theta}=2^{n-1}\cos\theta\left(\cos^2\theta-\cos^2\frac{\pi}{n}\right)\left(\cos^2\theta-\cos^2\frac{2\pi}{n}\right).....$$

$$\times\left(\cos^2\theta-\cos^2\frac{n-2}{2n}\pi\right)$$

$$=2^{n-1}\cos\theta\left(\sin^2\frac{\pi}{n}-\sin^2\theta\right)\left(\sin^2\frac{2\pi}{n}-\sin^2\theta\right).....$$

$$\times\left(\sin^2\frac{n-2}{2}\frac{\pi}{n}-\sin^2\theta\right) \quad ... \quad (13)$$

If we now allow $\theta\to 0$, we obtain

$$n=2^{n-1}\sin^2\frac{\pi}{4}\sin^2\frac{2\pi}{n}.....\sin^2\frac{n-1}{2}\frac{\pi}{n}, \quad ... \quad (14)$$

when n is odd; and

$$n=2^{n-1}\sin^2\frac{\pi}{n}\sin^2\frac{2\pi}{n}.....\sin^2\frac{n-2}{2}\frac{\pi}{n}, \quad ... \quad (15)$$

when n is even.

Now dividing (12) by (14) and (13) by (15), we obtain

$$\frac{\sin n\theta}{n\sin\theta}=2^{n-1}\left(1-\frac{\sin^2\theta}{\sin^2\frac{\pi}{n}}\right)\left(1-\frac{\sin^2\theta}{\sin^2\frac{2\pi}{n}}\right)\left(1-\frac{\sin^2\theta}{\sin^2\frac{n-2}{2}\frac{\pi}{n}}\right)... \quad (16)$$

when n is odd; and

$$\frac{\sin n\theta}{n\sin\theta\cos\theta}=2^{n-1}\left(1-\frac{\sin^2\theta}{\sin^2\frac{\pi}{n}}\right)\left(1-\frac{\sin^2\theta}{\sin^2\frac{2\pi}{n}}\right).....\left(1-\frac{\sin^2\theta}{\sin^2\frac{n-1}{2}\frac{\pi}{n}}\right),$$

when n is even. ... (17)

Cor. Since all the angles involved in (14) and (15) are acute,

$$\sqrt{n}=2^{\frac{n-1}{2}}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}.....\sin\frac{n-1}{2}\frac{\pi}{n}, \quad ... \quad (18)$$

when n is odd; and

$$\sqrt{n}=2^{\frac{n-1}{2}}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\sin\frac{3\pi}{n}.....\sin\frac{n-2}{2}\cdot\frac{\pi}{n}, \quad (19)$$

when n is even.

Ex. Prove that

$$2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}.....\sin\frac{(n-1)\pi}{n}=n. \quad (P.\ U.\ 1941)$$

11·3. Binomial Equations. Factors of Binomial Expressions. Roots of +1 and −1.

In the present article we consider the four binomial expressions,

$$x^{2m}+1,\ x^{2m+1}+1,\ x^{2m}-1,\ x^{2m+1}-1,$$

and the corresponding equations obtained by equating them to zero.

11·31. First equation $x^{2m}+1=0$.

The equation is a standard reciprocal equation.

$$x^{2m}=-1=\cos(2p+1)\pi+i\sin(2p+1)\pi,$$

where p is a $+ve$ integer.

$$\therefore\quad x=\{\cos(2p+1)\pi+i\sin(2p+1)\pi\}^{\frac{1}{2m}}$$

$$=\cos\frac{(2p+1)\pi}{2m}+i\sin\frac{(2p+1)\pi}{2m},\quad p=0,\ 1,\ 2,\ldots\overline{2m-1}.$$

$$=\cos\frac{(2p+1)\pi}{2m}\pm i\sin\frac{(2p+1)\pi}{2m},\quad p=0,\ 1,\ 2,\ldots\overline{m-1}$$

$$\because\quad \operatorname{cis}\frac{(2p-k)\pi}{p}=\cos\frac{k\pi}{p}-i\sin\frac{k\pi}{p}.$$

These roots are all distinct, and being $2m$ in number there is no other root.

This also follows from the fact that the roots occur in reciprocal pairs. Thus the factors of $x^{2m}+1$ are of the form

$$\left\{x-\cos\frac{2p+1}{2m}\pi-i\sin\frac{2p+1)\pi}{2m}\right\}\left\{x-\cos\frac{2p+1}{2m}\pi+i\sin\frac{2p+1}{2m}\pi\right\}$$

$$=x^2-2x\cos\frac{2p+1}{2m}+1,\ p=0,\ 1,\ 2,\ldots\overline{m-1}.$$

Hence
$$x^{2m}+1=\prod_{1}^{m-1}\left(x^2-2x\cos\frac{2p+1}{2m}+1\right),\qquad(20)$$

and these are the only factors, as they are m in number and are all distinct.

The result may be written as

$$x^m+x^{-m}=\prod_{1}^{m-1}\left((x+x^{-1}-2\cos\frac{2p+1}{2m}\pi\right).$$

If in this result we put $x=\cos\theta+i\sin\theta$, we get the factors of $\cos n\theta$ (Art. 11·2)

11·32. Equation $x^{2m+1}+1=0$.

One root is obviously -1. The other roots are determined as before.

Here $x=(-1)^{\frac{1}{2m+1}}=\{\cos(2p+1)\pi+i\sin(2p+1)\pi\}^{\frac{1}{2m+1}}$

$$=\cos\frac{(2p+1)\pi}{2m+1}+i\sin\frac{(2p+1)\pi}{2m+1},\quad p=0,\ 1,\ 2,\ldots\overline{2m}.$$

These roots are all different, and being $2m+1$ in number, there is no other root.

For $p=m$, we get the root -1 already found, and the other roots occur in reciprocal pairs like

$$\cos\frac{(2p+1)\pi}{2m+1}+i\sin\frac{(2p+1)\pi}{2m+1},\quad \cos\frac{(2p+1)\pi}{2m+1}-i\sin\frac{(2p+1)\pi}{2m+1}.$$

These roots give the quadratic factors like

$$x^2-2x\cos\frac{2p+1}{2m+1}\pi+1,\ p=0,\ 1,\ 2,\ldots\overline{m-1}.$$

Hence $$x^{2m+1}+1=(x+1)\prod_{p=0}^{m-1}(x^2-2x\cos\frac{2p+1}{2m+1}\pi+1). \qquad (21)$$

The fractorization is unique. For there cannot be more than $2m+1$ factors, and also all these factors are different.

11·33 Equation $x^{2m+1}-1=0$.

This is a reciprocal equation of the second kind and odd degree. Obviously $+1$ is a root of the equation.

Here $x=(1)^{\frac{1}{2m+1}}=[\cos 2p\pi+i\sin 2p\pi]^{\frac{1}{2m+1}}$

$$=\cos\frac{2p\pi}{2m+1}+i\sin\frac{2p\pi}{2m+1},\quad p=0,\ 1,\ldots 2m.$$

For $p=0$ is obtained the root 1. The other values of p, equidistant from beginning and end, give reciprocal roots like

$$\cos\frac{2p\pi}{2m+1}+i\sin\frac{2p\pi}{2m+1},\quad \cos\frac{2p\pi}{2m+1}-i\sin\frac{2p\pi}{2m+1},$$

and such roots give the real quadratic factors

$$x^2-2x\cos\frac{2p\pi}{2m+1}+1,\ p=1,\ 2,\ldots m.$$

$$\therefore\quad x^{2m+1}-1=(x-1)\prod_{1}^{m}(x^2-2x\cos\frac{2p\pi}{2m+1}+2)\ldots\ldots \qquad (22)$$

This factorisation is unique.

11·34 Equation $x^{2m}-1=0$.

Here $x=(1)^{\frac{1}{2m}}=(\cos 2p\pi+i\sin 2p\pi)^{\frac{1}{2m}}$, [$p$ is a $+ve$ integer].

$$=\cos\frac{2p\pi}{2m}+i\sin\frac{2p\pi}{2m},\quad p=0,\ 1,\ 2,\ldots\ldots\overline{2m-1}.$$

For $p=0$ and $p=m$ we get the roots 1 and -1. The other roots occur as reciprocal pairs.

$$x=\cos\frac{2p\pi}{2m}\pm i\sin\frac{2p\pi}{2m},\quad p=1, 2,\ldots\ldots \overline{m-1};$$

and these are the only roots, for no other root is obtained from any other value of p.

Hence $$x^{2m}-1=(x^2-1)\prod_{1}^{m-1}(x^2-2x\cos\frac{p\pi}{m}+1). \quad \ldots (23)$$

11·4. Factors of $x^{2n}-2x^n\cos n\theta+1$. The factors of this expression can be obtained by a method similar to that of Art. 11·3. But it would be interesting to give an alternative method.

Let $u_n\equiv x^n-2\cos n\theta+x^{-n}$.

Now $$x^n-2\cos n\theta+x^{-n}=(x^{n-1}+x^{-n+1})(x-2\cos\theta+x^{-1})$$
$$+2\cos\theta\{x^{n-1}-2\cos(n-1)\theta+x^{-n+1}\}$$
$$-\{x^{n-2}-2\cos(n-2)\theta+x^{-n+2}\}.$$

$\therefore\ u_n=(x^{n-1}+x^{-n+1})(x-2\cos\theta+x^{-1})+2\cos\theta\, u_{n-1}-u_{n-2}$.

Thus u_n is divisible by u_1, if u_{n-1} and u_{n-2} are divisible by u_1.

But $$u_2=x^2-2\cos 2\theta+x^{-2}$$
$$=(x-2\cos\theta+x^{-1})(x+2\cos\theta+x^{-1}).$$

Hence u_2 is divisible by u_1 and, therefore, u_3 is also divisible, and so on. Hence u_n is divisible by u_1 and therefore $x^{2n}-2x^n\cos n\theta+1$ has the factor $x^2-2x\cos\theta+1$. Since θ can be changed into $\theta+\frac{2p\pi}{n}$ without altering $\cos n\theta$, we see that if p is a positive integer,

$$x^2-2x\cos\left(\theta+\frac{2p\pi}{n}\right)+1$$

is a factor of the given expression.

$$\therefore\ x^{2n}-2x^n\cos n\theta+1=\prod_{p=0}^{n-1}\left\{[x^2-2x\cos\left(\theta+\frac{2px}{n}\right)+1\right\}. \quad \ldots(24)$$

Cor. i. $$x^n-2\cos n\theta+x^{-n}=\prod_{p=0}^{n-1}[x-2\cos(\theta+\frac{2p\pi}{n})+x^{-1}]. \quad \ldots(25)$$

If we put $x=\cos\varphi+i\sin\varphi$ we obtain

$$\cos n\varphi-\cos n\theta=2^{n-1}\prod_{p=0}^{p=n-1}\left\{\cos\varphi-\cos\left(\theta+\frac{2p\pi}{n}\right)\right\}\ldots(26)$$

Cor. 2. If x be replaced by $\frac{x}{a}$, then multiphying by a^{2n},

$$x^{2n}-2a^nx^n\cos n\theta+a^{2n}=\prod_{p=0}^{n-1}\left[x^2-2ax\cos\left(\theta+\frac{2p\pi}{n}\right)+a^2\right] \quad ...(27)$$

*11·5. Expansion of cos nθ and sin nθ in powers of cos θ, n being a positive integer.

In art. 10·41 we have shown the possibility of expressing $\cos n\theta$ and $\sin n\theta/\sin\theta$ as polynomials of degrees n and $n-1$, respectively. We can accordingly assume

$$\cos n\theta=A_0x^n+A_2x^{n-2}+...+A_{2r}x^{n-2r}+...$$

$$\frac{\sin n\theta}{\sin\theta}=B_1x^{n-1}+B_3x^{n-3}+...+B_{2r+1}x^{n-2r-1}+...,$$

where $x=\cos\theta$, and A's and B's are constants which are to be determined.

Let $y=\cos n\theta$, or $\cos^{-1}y=n\theta$,

$$\therefore\ \cos^{-1}y=n\cos^{-1}x.$$

Differentiating w.r. to x,

$$\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}=\frac{n}{\sqrt{1-x^2}}.$$

$$\therefore\quad (1-x^2)\left(\frac{dy}{dx}\right)^2=n^2(1-y)^2$$

Differentiating again,

$$2(1-x^2).\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}-2x\left(\frac{dy}{dx}\right)^2=-2n^2y\frac{dy}{dx},$$

or
$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+n^2y=0 \quad (1)$$

Since $y=A_0x^n+A_2x^{n-2}+...+A_{2r}x^{n-2r}+...$

$$\therefore\ \frac{dy}{dx}=nA_0x^{n-1}+(n-2)A_2x^{n-3}+...+(n-2r)A_{2r}x^{n-2r-1}+...$$

$$\frac{d^2y}{dx^2}=n(n-1)A_0x^{n-2}+(n-2)(n-3)A_2x^{n-4}+...$$

$$+(n-2r)(n-2r-1)A_{2r}x^{n-2r-2}+...$$

Substituting these values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the differential equation (1), we obtain an identity; consequently the coefficients of the various powers of x must vanish.

$$\therefore\ n(n-1)A_0-(n-2)(n-3)A_2-(n-2)A_2+n^2A_2=0.$$

$$\therefore\ A_2=-\frac{n}{4}A_0.$$

In general equating to zero the coefficient of x^{n-2r}

$$(n-2r+2)(n-2r+1)A_{r-2}-(n-2r)(n-2r-1)A_{2r}$$
$$-(n-2r)A_{2r}+n^2A2_r=0,$$

or $$(n-2r+2)(n-2r+1)A_{r-2}=-4r(n-r)A_{2r},$$

or $$A_{2r}=-\frac{(n-2r+2)(n-2r+1)}{4r(n-r)}A_{2r-2} \quad \ldots (2)$$

Now $$A_0=2^{n-1}, \quad \therefore\ A_2=-n\,2^{n-3},$$

$$A_4=-\frac{(n-2)(n-3)}{4.2(n-2)}\ A_2=\frac{n(n-3)}{2!}2^{n-5},$$

$$A_6=-\frac{(n-4)(n-5)}{4.3.(n-3)}A_4=-\frac{n(n-4)(n-5)}{3!}2^{n-7}.$$

We may now assume that

$$A_{2r}=(-1)^r\frac{n(n-r-1)(n-r-2)\ldots(n-2r+1)}{r!}2^{n-2r-1}.$$

From (2), $$A_{2r+2}=-\frac{(n-2r)(n-2r-1)}{4(r+1)(n-r-1)}A_{2r}$$

$$=(-1)^{r+1}\frac{n(n-r-2)(n-r-3)\ldots(n-2r)(n-2r-1)}{(r+1)!}2^{n-2r-3}.$$

So if the law of formation of A_{2r} holds for any value of r, it also holds for $r+1$. But it holds for $r=1, 2, 3$; hence for any integral value of r.

$\therefore\ 2\cos n\theta$

$$=(2c)^n-\frac{n}{1!}(2c)^{n-2}+\frac{n(n-3)}{2!}(2c)^{n-4}-\frac{n(n-4)(n-5)}{3!}.(2c)^{n-6}$$
$$+\ldots+(-1)^r\frac{n(n-r-1)(n-r-2)\ldots(n-2r+1)}{r!}(2c)^{n-2r}+\ldots$$

The expansion of $\sin n\theta/\sin\theta$ can be obtained by differentiating the above result. It will be found that

$$\frac{\sin n\theta}{\sin\theta}=(2c)^{n-1}-\frac{n-2}{1!}(2c)^{n-3}+\frac{(n-3)(n-4)}{2!}(2c)^{n-5}$$
$$-\frac{(n-4)(n-5)(n-6)}{3!}(2c)^{n-7}+\ldots.$$
$$+(-1)r\frac{(n-r-1)(n-r-2)..(n-2r)}{r!}(2c)^{n-2r-1}+\ldots(4)$$

Examples XI (*b*)

Solve the following equations :—

1. $x^7+1=0.$
2. $x^7+x^4+x^3+1=0.$
3. $x^4-x^3+x^2-x+1=0.$
4. Solve the equation $x^{12}-1=0$, and find which of its roots satisfy the equation $x^4+x^2+1=0.$

[Solution. $x^{12}=1=\cos 2p\pi+i\sin 2p\pi$.

$$\therefore \quad x=\cos\frac{p\pi}{6}\pm i\sin\frac{p\pi}{6},\quad p=0,\ 1,\ 2,\ 3,\ 4,\ 5.$$

Suppose that $\cos\frac{p\pi}{6}+i\sin\frac{p\pi}{6}$ is a root of the equation.

$$\therefore \quad \cos\frac{4p\pi}{6}+i\sin\frac{4p\pi}{6}+\cos\frac{2p\pi}{6}+i\sin\frac{2p\pi}{6}+1=0.$$

$$\therefore \quad \cos\frac{2p\pi}{3}+\cos\frac{p\pi}{3}+1=0, \qquad \ldots \text{ (A)}$$

$$\sin\frac{2p\pi}{3}+\sin\frac{p\pi}{3}=0. \qquad \ldots \text{ (B)}$$

From (B), $\sin\frac{p\pi}{3}=0$, or $\cos\frac{p\pi}{3}=-\frac{1}{2}$.

The solution $\sin\frac{p\pi}{2}=0$ is inconsistent with (A).

$$\therefore \quad \cos\frac{p\pi}{3}=-\tfrac{1}{2}, \text{ and } \cos\frac{2p\pi}{3}=-\tfrac{1}{2} \text{ from (A).}$$

Hence $p=2$, or 4.
Thus the common roots are

$$\cos\frac{\pi}{3}\pm i\sin\frac{\pi}{3};\ \cos\frac{2\pi}{3}\pm i\sin\frac{2\pi}{3}.]$$

5. Solve the equation $(1+x)^5+x^5=0$.

[Solution. $(1+x)^5=-x^5=x^5(\cos\overline{2r+1}\,\pi+i\sin\overline{2r+1}\,\pi)$,

$$\therefore \quad 1+x=x\left(\cos\frac{2r+1}{5}\pi+i\sin\frac{2r+1}{5}\pi\right).$$

$$\therefore \quad x\left[1-\cos\frac{2r+1}{5}\pi-i\sin\frac{2r+1}{5}\pi\right]=-1,$$

$$2x\sin\frac{2r+1}{10}\pi\left[\sin\frac{2r+1}{10}\pi-i\cos\frac{2r+1}{10}\pi\right]=-1,$$

$$\text{or} \quad 2x\sin\frac{2r+1}{10}\pi=-\left[\sin\frac{2r+1}{10}\pi+i\cos\frac{2r+1}{10}\pi\right].$$

$$\therefore \quad x=-\tfrac{1}{2}\left[1+i\cot\frac{2r+1}{10}\pi\right],\ r=0,\ 1,\ 2,\ 3,\ 4.$$

6. Prove that every root of $(1+x)^6+x^6=0$ has $-\frac{1}{2}$ for its real part.

7. Show that the roots of the equation $(x-1)^n=x^n$ are

$$\tfrac{1}{2}\left(1+i\cot\frac{\pi r}{n}\right),\ r=0,\ 1,\ 2,\ldots\ldots(n-1).$$

8. Solve the equation $(5+x)^5+(5-x)^5=0$.

[Sol. $(5+x)^5=-(5-x)^5=(5-x)^5(\cos\overline{2r+1}\,\pi+i\sin\overline{2r+1}\,\pi)$

Extracting the fifth root,

$$(5+x)=(5-x)\left(\cos\frac{2r+1}{5}\pi+i\sin\frac{2r+1}{5}\pi\right)\ [r=0,\ 1,\ 2,\ 3,\ 4$$

$$\therefore\quad x\left[1+\cos\frac{2r+1}{5}\pi+i\sin\frac{2r+1}{5}\pi\right],$$

$$=5\left[\cos\frac{2r+1}{5}\pi+i\sin\frac{2r+1}{5}\pi-1\right],$$

or
$$x\left[\cos\frac{2r+1}{10}\pi+i\sin\frac{2r+1}{10}\pi\right]\cos\frac{2r+1}{10}\pi$$

$$=-5\left[\sin\frac{2r+1}{10}\pi-i\cos\frac{2r+1}{10}\pi\right]\sin\frac{2r+1}{10}\pi$$

$$=5i\left[\cos\frac{2r+1}{10}\pi+i\sin\frac{2r+1}{10}\pi\right]\sin\frac{2r+1}{10}\pi,$$

Hence $x=5i\tan\dfrac{2r+1}{10}\pi,\qquad r=0,\ 1,\ 2,\ 3,\ 4]$.

9. Prove that the roots of the equation $(x+i)^6+(x-i)^6=0$ are

$$\cot\frac{2r+1}{12}\pi\qquad r=0,\ 1,\ 2,\ 3,\ 4,\ 5.$$

10. Show that the equation $(x+1)^6+(x-1)^6=0$ has

$i\cot\dfrac{2r+1}{12}\pi$, $r=0,\ 1,\ 2,\ 3,\ 4,\ 5$ for its roots.

Hence show that

$$(x+1)^6+(x-1)^6$$

$$=2\left(x^2+\cot^2\frac{\pi}{12}\right)\left(x^2+\cot^2\frac{3\pi}{12}\right)\left(x^2+\cot^2\frac{5\pi}{12}\right)$$

$$=2\left(x^2+\tan^2\frac{\pi}{12}\right)\left(x^2+\tan^2\frac{3\pi}{12}\right)\left(x^2+\tan^2\frac{5\pi}{12}\right).$$

Deduce $\sec^2\dfrac{\pi}{12}+\sec^2\dfrac{3\pi}{12}+\sec^2\dfrac{5\pi}{12}=18.$

11. Show that the solutions of $(1+x)^{2n}+(1-x)^{2n}=0$ are

$$x=\pm i\tan\frac{2r+1}{4n},\quad r=0,\ 1,\ 2,\ldots\overline{n-1}.$$

Deduce $(1+x)^{2n}+(1-x)^{2n}=2\prod_{0}^{n-1}\left(x^2+\tan^2\dfrac{2r+1}{4n}\pi\right)$.

Hence show that $\sum_{0}^{n-1}\sec^2\dfrac{2r+1}{4n}\pi=2n^2$.

12. Solve the equation $z^3=i(z-1)^3$ and show that the points on Argand's diagram corresponding to these roots are collinear.

[The points are $z=\frac{1}{2}-\frac{1}{2}\,i\tan\frac{4r+1}{12}\pi$, $r=0, 1, 2$, and they lie on the line $x=\frac{1}{2}$],

13. Solve the equation $(1+iz)^3=-1$, and show that the three points on Argand's diagram corresponding to these roots are the vertices of an equilateral triangle.

[The points are $2i$, $\frac{1}{2}(i\pm\sqrt{3})$]

14. Show that the solutions of the equation

$$(x+1)^7=x^7(\cos 7a+i\sin 7a)$$

are $-\frac{1}{2}-\frac{1}{2}\,i\cot\left(\frac{1}{2}a+\frac{p\pi}{7}\right)$, $p=0, 1, 2, 3, 4, 5, 6$]

15. Prove that

$$x^7+1=(x+1)\left(x^2-2x\cos\frac{\pi}{7}+1\right)\left(x^2-2x\cos\frac{3\pi}{7}+1\right)$$
$$\times\left(x^2-2x\cos\frac{5\pi}{7}+1\right).$$

Deduce $\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}=4\sin\frac{\pi}{14}\cdot\sin\frac{3\pi}{14}\cdot\sin\frac{5\pi}{14}=\frac{1}{2}$

[Equate the coefficient of x^6. For the second half put $x=1$].

16. Show that $x^{10}+1=\prod_{r=0}^{4}\left\{(x-1)^2+4x\sin^2\frac{2r+1}{20}\pi\right\}$

Deduce (i) $\sin\frac{\pi}{20}\sin\frac{3\pi}{20}\sin\frac{5\pi}{20}\sin\frac{7\pi}{20}\sin\frac{9\pi}{20}=\frac{\sqrt{2}}{32}$,

(ii) $\sin\frac{\pi}{10}\sin\frac{3\pi}{10}\sin\frac{5\pi}{10}\sin\frac{7\pi}{10}\sin\frac{9\pi}{10}=\frac{1}{16}$.

[For (i) put $x=1$, and extract the square root. For (ii) put $x=-1$, and extract the square-root and then multiply by (i)].

17. If a be a complex root of $x^{13}=1$, find the cubic whose roots are $a+a^5+a^8+a^{12}$, $a^2+a^{10}+a^3+a^{11}$, $a^4+a^6+a^9+a^7$.

[Solution. The sum of the roots is -1, therefore the first two terms in the equation are of the form z^3+z^2. Calling any one root (say the first) to be z, we have

$$z=a+\frac{1}{a}+a^5+\frac{1}{a^5}.$$

$$\therefore \quad z^2=\alpha^2+\frac{1}{\alpha^2}+\alpha^{10}+\frac{1}{\alpha^{10}}+2\left(\alpha^6+\frac{1}{\alpha^6}+\alpha^4+\frac{1}{\alpha^4}+2\right)$$

$$=\alpha+\frac{1}{\alpha}+\alpha^2+\frac{1}{\alpha^2}+\alpha^3+\frac{1}{\alpha^3}+\alpha^4+\frac{1}{\alpha^4}+\alpha^5+\frac{1}{\alpha^5}+\alpha^6+\frac{1}{\alpha^6}$$
$$-\left(\alpha+\frac{1}{\alpha}+\alpha^5+\frac{1}{\alpha^5}\right)+4+\left(\alpha^4+\frac{1}{\alpha^4}+\alpha^6+\frac{1}{\alpha^6}\right)$$

$$=3-z+\alpha^4+\frac{1}{\alpha^4}+\alpha^6+\frac{1}{\alpha^6}.$$

$$\therefore \quad z(z^2+z-3)=\left(\alpha+\frac{1}{\alpha}+\alpha^5+\frac{1}{\alpha^5}\right)\left(\alpha^4+\frac{1}{\alpha^4}+\alpha^6+\frac{1}{\alpha^6}\right)$$

$$=2\alpha^5+\frac{1}{\alpha^3}+\alpha^7+2\frac{1}{\alpha^5}+\alpha^3+\frac{1}{\alpha^7}+\alpha^9+2\alpha+\alpha^{11}+2\frac{1}{\alpha}+\frac{1}{\alpha^9}+\frac{1}{\alpha^{11}}$$

$$=-1+z.$$

Thus the required equation is

$$z^3+z^2-4z+1=0.$$

Second Solution : Call the roots z_1, z_2, z_3; then $z_1+z_2+z_3=-1$.

Also it is easily verified that

$$z_2z_3=-1+z_3,\ z_3z_1=-1+z_1,\ z_1z_2=-1+z_2,$$
$$\therefore \quad \Sigma z_1z_2=-3+\Sigma z_1=-4.$$

Multiplying these relations by z_1, z_2, z_3, and adding, we obtain

$$3z_1z_2z_3=-\Sigma z_1+\Sigma z_2z_3=1-4.$$
$$\therefore \quad z_1z_2z_3=-1.$$

Hence the required equation, is $z^3-z^2\Sigma z_1+z\Sigma z_1z_2-z_1z_2z_3=0$,

i.e., $$z^3+z^2-4z+1=0.$$

The equation $x^{13}-1=0$ can now be solved algebraically. The roots z_1, z_2, z_3, are all real and can be found by Horner's method.

Also $\alpha+\frac{1}{\alpha}$, $\alpha^5+\frac{1}{\alpha^5}$ are the roots of $t^2-z_1t+z_3=0$;

$\alpha^2+\frac{1}{\alpha^2}$, $\alpha^3+\frac{1}{\alpha^3}$ of $t^2-z_2t+z_1=0$; and $\alpha^4+\frac{1}{\alpha^4}$, $\alpha^6+\frac{1}{\alpha^6}$

of $t^2-tz_3+z_2=0$. When these quadratics are solved, α, $\frac{1}{\alpha}$; α^2, $\frac{1}{\alpha^2}$;

...may be found from quadratics.]

18. Prove that the roots of the equation $x^{10}+11x^5-1=0$

are $$\frac{\pm\sqrt{5}-1}{2}\left\{\cos\frac{2p\pi}{5}+i\ \sin\frac{2p\pi}{5}\right\}.$$

CHAPTER XII

EXPONENTIAL AND HYPERBOLIC FUNCTIONS

12·1. Exponential function of a real variable.

The infinite series

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+\ldots$$

converges for all values of x. (Examples : VII(d) 11). If E (x) denotes the sum of the series, the function E (x) is finite and single-valued.

Since E (x), E (y) are both absolutely convergent, therefore by the multiplication theorem of the infinite series (7·931), E (x) E (y)

$$=\left(1+x+\frac{x^2}{2!}+..+\frac{x^n}{n!}+\ldots\right)\left(1+\frac{y}{1!}+\frac{y^2}{2!}+\ldots+\frac{y^n}{n!}+\ldots\right)$$

$$=1+(x+y)+\left(\frac{x^2}{2!}+\frac{xy}{1!}+\frac{y^2}{2!}\right)+\ldots$$

$$+\left(\frac{x^n}{n!}+\frac{x^{n-1}}{(n-1)!}\cdot\frac{y}{1!}+\frac{x^{n-2}}{(n-2)!}\cdot\frac{y^2}{2!}+\ldots+\frac{x}{1!}\cdot\frac{y^{n-1}}{(n-1)!}+\frac{y^n}{n!}\right)+\ldots$$

Now $\frac{x^n}{n!}+\frac{x^{n-1}}{(n-1)!}\cdot\frac{y}{1!}+\frac{x^{n-2}}{(n-2)!}\cdot\frac{y^2}{2!}+\ldots+\frac{x}{1!}\cdot\frac{y^{n-1}}{(n-1)!}+\frac{y^n}{n!}$

$$=\frac{1}{n!}\left(x^n+{}^nC_1x^{n-1}y+{}^nC_2x^{n-2}y^2+\ldots+{}^nC_{n-1}xy^{n-1}+{}^nC_ny^n\right)$$

$$=\frac{1}{n!}(x+y)^n.$$

Hence E (x) E (y) $=1+(x+y)+\frac{(x+y)^2}{2!}+\ldots\frac{(x+y)^n}{n!}+\ldots$

$$=\text{E}(x+y). \tag{1}$$

Thus by repetition of the above formula

$$\text{E}(x)\,\text{E}(y)\,\text{E}(z)\ldots=\text{E}(x+y+z+\ldots). \tag{2}$$

In particular, if $x=y=z=\ldots$, and supposing the number of letters to be n,

$$\{\text{E}(x)\}^n=\text{E}(nx). \tag{3}$$

If $x=1$, we get the particular case

$$\{\text{E}(1)\}^n=\text{E}(n). \tag{4}$$

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In the formula (3) above, suppose x is the positive rational $\frac{m}{n}$, then m and n being positive integers,

$$\left[\mathrm{E}\left(\frac{m}{n}\right)\right]^n = \mathrm{E}(m) = [\mathrm{E}(1)]^m. \tag{5}$$

Thus $\mathrm{E}\left(\frac{m}{n}\right)$ is a nth root of $[\mathrm{E}(1)]^m$. Since m and n are positive integers, the value of $\mathrm{E}\left(\frac{m}{n}\right)$ is real and positive.

Thus the nth root of $\mathrm{E}(m)$ or $[\mathrm{E}(1)]^m$ is a positive real number.

If in the formula (1), we put $y=-x$, x being a positive rational,

$$\mathrm{E}(x)\,\mathrm{E}(-x)=\mathrm{E}(0)=1,$$

so that
$$\mathrm{E}(-x)=[\mathrm{E}(x)]^{-1}=[\mathrm{E}(1)]^{-x}. \tag{6}$$

Thus (4), (5), and (6) lead to the conclusion, that for all rational values of x

$$\mathrm{E}(x)=[\mathrm{E}(1)]^x=e^x, \tag{7}$$

where
$$e=\mathrm{E}(1)=1+\frac{1}{1!}+\frac{1}{2!}+\ldots$$

Finally, we define e^x, *when* x *is irrational, as being equal to* $E(x)$.

12·12. Differentiation of e^x.

We first prove the following lemma :

$$\operatorname*{Lt}_{x\to 0}\frac{e^x-1}{x}=1.$$

$$\frac{e^x-1}{x}=1+\frac{x}{2!}+\frac{x^2}{3!}+\ldots$$

$$<1+\left|\frac{x}{2}\right|+\left|\frac{x}{2}\right|^2+\left|\frac{x}{2}\right|^3+\ldots$$

$$=1+\frac{\left|\frac{x}{2}\right|}{1-\left|\frac{x}{2}\right|}.$$

Now $\left|\frac{x}{2}\right| \Big/ \left[1-\left|\frac{x}{2}\right|\right] \longrightarrow 0$, as $x \longrightarrow 0$,

hence
$$\operatorname*{Lt}_{x\to 0}\frac{e^x-1}{x}=1.$$

Now
$$\mathrm{D}_x e^x=\operatorname*{Lt}_{h\to 0}\frac{e^{x+h}-e^x}{h}=e^x\operatorname*{Lt}_{h\to 0}\frac{e^h-1}{h}=e^x. \tag{1}$$

Incidentally we have proved that e^x is a continuous function of x.

12·13. The three fundamental operations of arithmetic on the functions e^x and e^{-x} give new functions which are called **hyperbolic functions**, and are defined as follows :—

$$\cosh x=\frac{e^x+e^{-x}}{2} \quad (1) \qquad \sinh x=\frac{e^x-e^{-x}}{2} \quad (2)$$

$$\operatorname{sech} x=\frac{2}{e^x+e^{-x}} \quad (3) \qquad \operatorname{cosech} x=\frac{2}{e^x-e^{-x}} \quad (4)$$

$$\tanh x=\frac{e^x-e^{-x}}{e^x+e^{-x}} \quad (5) \qquad \coth x=\frac{e^x+e^{-x}}{e^x-e^{-x}} \quad (6)$$

These are, called, respectively, *hyperbolic cosine*, *hyperbolic sine* etc.

Examples XII (a)

1. Show that cosh x and sech x are even functions but, tanh x, coth x, sech x and cosech x are all odd functions of x.

2. Prove that

$$\sinh x=1+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\cdots \quad (7)$$

$$\cosh x=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots \quad (8)$$

3. Show that $\tanh x=\dfrac{\sinh x}{\cosh x}$, $\coth x=\dfrac{\cosh x}{\sinh x}$.

$$\operatorname{sech} x=\frac{1}{\cosh x}. \quad \operatorname{cosech} x=\frac{1}{\sinh x},$$

4. Prove that $\cosh 0=1$, $\sinh 0=0$,
$\operatorname{sech} 0=1$, $\tanh 0=0$.

5. Prove that $\underset{x\to 0}{\mathrm{Lt}}\dfrac{e^{ax}-1}{\sin bx}=\dfrac{a}{b}$.

12·21. Fundamental Relations between the hyperbolic functions of the same argument.

We shall prove the following relations

$$\cosh^2 x-\sinh^2 x=1, \quad (1)$$

$$\operatorname{sech}^2 x+\tanh^2 x=1, \quad (2)$$

$$\coth^2 x-\operatorname{cosech}^2 x=1. \quad (3)$$

By definition $\cosh x+\sinh x=e^x$,

$\cosh x-\sinh x=e^{-x}$.

$\therefore$ $\cosh^2 x-\sinh^2 x=e^x.e^{-x}=1$.

Dividing both sides of (1) by $\cosh^2 x$, we get

$$1-\frac{\sinh^2 x}{\cosh^2 x}=\frac{1}{\cosh^2 x},$$

or

$$1-\tanh^2 x=\operatorname{sech}^2 x,$$

which is equivalent to (2). To get (3) we divide both sides of (1) by $\sinh^2 x$.

Note. Relation (2) shows that $-1 < \tanh x < 1$.

Since by definition, $\cosh x$ is always positive, it follows that so is $\operatorname{sech} x$. The relations (1) and (3), therefore, show that

$$\cosh x \geqslant 1,\ 0 < \operatorname{sech} x \leqslant 1.$$

12·211. These correspond to the relations

$$\cos^2\theta + \sin^2\theta = 1,\ \sec^2\theta - \tan^2\theta = 1,\ \operatorname{cosec}^2\theta - \cot^2\theta = 1$$

between the circular functions. From any formula connecting the circular functions, the corresponding formula for hyperbolic functions may be deduced by the following rule due to Osborne.

In any formula connecting the circular functions of general angles, replace each circular function by the corresponding hyperbolic function and change the sign of every product (or implied product) of two sines.

This is illustrated by (1).

In $\sec^2\theta - \tan^2\theta = 1$, since $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, $\dfrac{\sin^2\theta}{\cos^2\theta}$ will be replaced by $-\dfrac{\sinh^2\theta}{\cosh^2\theta} = -\tanh^2\theta$.

Hence the relations (2) and (3).

By means of relations (1), (2), (3) combined with the definitions, any hyperbolic function can be expressed in terms of any other.

The following table gives the results:—

	$\sinh x = u$	$\cosh x = u$	$\tanh x = u$	$\coth x = u$	$\operatorname{sech} x = u$	$\operatorname{cosech} x = u$
$\sinh x$	u	$\sqrt{u^2-1}$	$\dfrac{u}{\sqrt{1-u^2}}$	$\dfrac{1}{\sqrt{u^2-1}}$	$\dfrac{\sqrt{1-u^2}}{u}$	$\dfrac{1}{u}$
$\cosh x$	$\sqrt{1+u^2}$	u	$\dfrac{1}{\sqrt{1-u^2}}$	$\dfrac{u}{\sqrt{u^2-1}}$	$\dfrac{1}{u}$	$\dfrac{\sqrt{1+u^2}}{u}$
$\tanh x$	$\dfrac{u}{\sqrt{1+u^2}}$	$\dfrac{\sqrt{u^2-1}}{u}$	u	$\dfrac{1}{u}$	$\sqrt{1-u^2}$	$\dfrac{1}{\sqrt{1+u^2}}$

	$\sin x=u$	$\cosh x=u$	$\tanh x=u$	$\coth x=u$	$\operatorname{sech} x=u$	$\operatorname{cosech} x=u$
$\coth x$	$\frac{\sqrt{1+u^2}}{u}$	$\frac{u}{\sqrt{u^2-1}}$	$\frac{1}{u}$	u	$\frac{1}{\sqrt{1-u^2}}$	$\sqrt{1+u^2}$
$\operatorname{sech} x$	$\frac{1}{\sqrt{1+u^2}}$	$\frac{1}{u}$	$\sqrt{1-u^2}$	$\frac{\sqrt{u^2-1}}{u}$	u	$\frac{u}{\sqrt{1+u^2}}$
$\operatorname{cosech} x$	$\frac{1}{u}$	$\frac{1}{\sqrt{u^2-1}}$	$\frac{\sqrt{1-u^2}}{u}$	$\sqrt{u^2-1}$	$\frac{u}{\sqrt{1-u^2}}$	u

12·22. Addition and Subtraction formulae.

$$\cosh(x+y)=\cosh x\cosh y+\sinh x\sinh y, \quad (1)$$

$$\sinh(x+y)=\sinh x\cosh y+\cosh x\sinh y, \quad (2)$$

$$\tanh(x+y)=\frac{\tanh x+\tanh y}{1+\tanh x\,\tanh y}, \quad (3)$$

$$\cosh(x-y)=\cosh x\cosh y-\sinh x\sinh y, \quad (4)$$

$$\sinh(x-y)=\sinh x\cosh y-\cosh x\sinh y, \quad (5)$$

$$\tanh(x-y)=\frac{\tanh x-\tanh y}{1-\tanh x\,\tanh y} \quad (6)$$

To prove (1) by *definition* :

$$\cosh(x+y)=\tfrac{1}{2}(e^{x+y}+e^{-x-y})=\tfrac{1}{2}(e^x.e^y+e^{-x}.e^{-y})$$

$$=\tfrac{1}{2}[(\cosh x+\sinh x)(\cosh y+\sinh y)$$

$$+(\cosh x-\sinh x)(\cosh y-\sinh y)]$$

$$=\cosh x\cosh y+\sinh x\sinh y.$$

Again, $\sinh(x+y)=\tfrac{1}{2}[e^x.e^y-e^{-x}e^{-y}]$

$$=\tfrac{1}{2}[(\cosh x+\sinh x)(\cosh y+\sinh y)$$

$$-(\cosh x-\sinh x)(\cosh y-\sinh y)]$$

$$=\sinh x\cosh y+\cosh x\sinh y.$$

The formula (3) follows from (1) and (2) by division. Finally (4), (5), (6) can be deduced from (1), (2), (3), respectively, by changing y into $-y$.

12·23. Relations between Hyperbolic Functions of x and $2x$.

$$\cosh 2x=\cosh^2 x+\sinh^2 x \quad (1)$$

$$=2\cosh^2 x-1 \quad (2)$$

$$=1+2\sinh^2 x \quad (3)$$

$$=\frac{1+\tanh^2 x}{1-\tanh^2 x}. \quad (4)$$

$$\sinh 2x = 2\sinh x \cosh x \tag{5}$$

$$= \frac{2\tanh x}{1-\tanh^2 x}. \tag{6}$$

$$\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}. \tag{7}$$

Formulæ (1), (2), (3) follow from (1), (2), (3), 12·22 by putting $y=x$.

To get (2) and (3) we make use of (1) along with (12·21,1). To get (3) we note that

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$= \frac{\cosh^2 x+\sinh^2 x}{\cosh^2 x-\sinh^2 x} \qquad (12\cdot21,1)$$

$$= \frac{1+\tanh^2 x}{1-\tanh^2 x}. \quad \text{(dividing the top and bottom by } \cosh^2 x)$$

Again $\sinh 2x = 2\sinh x \cosh x$

$$= \frac{2\sinh x \cosh x}{\cosh^2 x-\sinh^2 x} \qquad (12\cdot21,1)$$

$$= \frac{2\tanh x}{1-\tanh^2 x}.$$

12·24. Hyperbolic Functions of $3x$. The following relations can be easily proved by the repeated application of art. (12·23).

$$\cosh 3x = 4\cosh^3 x - 3\cosh x, \tag{1}$$

$$\sinh 3x = 3\sinh x + 4\sinh^3 x, \tag{2}$$

$$\tanh 3x = \frac{3\tanh x+\tanh^3 x}{1+3\tanh^2 x}. \tag{3}$$

Examples XII (b)

1. Verify that the formulæ of arts. 12·21, 12·22, 12·23 may be deduced from the corresponding formulæ in cos x, sin x and tan x, by writing cosh x for cos x, i sinh x for sin x and i tanh x for tan x. (The reason for this will appear subsequently).

2. Prove the following relations :—

(*i*) $\sinh(x+y)+\sinh(x-y)=2\sinh x\cosh y$,
(*ii*) $\sinh(x+y)-\sinh(x-y)=2\cosh x\sinh y$,
(*iii*) $\cosh(x+y)+\cosh(x-y)=2\cosh x\cosh y$,
(*iv*) $\cosh(x+y)-\cosh(x-y)=2\sinh x\sinh y$.

3. Prove that

(*i*) $\sinh u+\sinh v=2\sinh\frac{u+v}{2}\cosh\frac{u-v}{2}$,

(*ii*) $\sinh u-\sinh v=2\cosh\frac{u+v}{2}\sinh\frac{u-v}{2}$,

(*iii*) $\cosh u+\cosh v=2\cosh\frac{u+v}{2}\cosh\frac{u-v}{2}$,

(*iv*) $\cosh u-\cosh v=2\sinh\frac{u+v}{2}\sinh\frac{u-v}{2}$.

4. Prove that $(\cosh x+\sinh x)^n=\cosh nx+\sinh nx$,
$(\cosh x-\sinh x)^n=\cosh nx-\sinh nx$,
n being a positive integer.

Proof. By definition
$\cosh x+\sinh x=e^x$, $\therefore$ $e^{nx}=\cosh nx+\sinh nx$
Hence $(\cosh x+\sinh x)^n=e^{nx}=\cosh nx+\sinh nx$.

If n is a positive integer, and $t=\tanh x$, show that

5. $\cosh nx=(\cosh x)^n\,[1+{}^nc_2\,t^2+{}^nc_4\,t^4+\ldots\ldots]$.

6. $\sinh nx=(\cosh x)^n\,[{}^nc_1t+{}^nc_3t^3+\ldots\ldots]$.

7. $\tanh nx=\dfrac{{}^nc_1t+{}^nc_3t^3+\ldots\ldots}{1+{}^nc_2t^2+{}^nc_4t^4+\ldots\ldots}$.

[Hint. $2\cosh nx=(\cosh x+\sinh x)^n+(\cosh x-\sinh x)^n$,
$2\sinh nx=(\cosh x+\sinh x)^n-(\cosh x-\sinh x)^n$].

8. If $\sin x\cosh y=\cos a$ and $\cos x\sinh y=\sin a$, prove that
$\sinh^2 y=\cos^2 x=\pm\sin a$.

9. Prove that $\coth\frac{x}{2}-\coth x=\operatorname{cosech} x$.

10. Prove that
$(\cosh x+\sinh x)(\cosh y+\sinh y)=\cosh(x+y)+\sinh(x+y)$.

11. Show that $\cosh^6 x-\sinh^6 x=1+\frac{3}{4}\sinh^2 2x$.

12. Show that $\left(\dfrac{1+\tanh x}{1-\tanh x}\right)^3=\cosh 6x+\sinh 6x$.

Prove the following :—

13. $\cosh 5x=16\,c^5-20c^3+5c$, $c\equiv\cosh x$.

14. $\sinh 7x=64\,s^7+112s^5+56s^3+7s$, $s\equiv\sinh x$.

15. $\cosh^6 x=\frac{1}{32}[\cosh 6x-6\cosh 4x+15\cosh 2x-10]$.

16. If $\tan x=\tan a\tanh\beta$, $\tanh y=\cot a\tanh\beta$, prove that
$\tan(x+y)=\sinh 2\beta\operatorname{cosec} 2a$.

17. If $\cosh u=\sec\theta$, where $-\pi<\theta<\pi$, and if $u\theta>0$, prove that $\sinh u=\tan\theta$, $u=\log(\sec\theta+\tan\theta)$, and $\tanh\frac{u}{2}=\tan\frac{\theta}{2}$.

[It $u\theta<0$, $\sinh u=-\tan\theta$, $u=\log(\sec\theta-\tan\theta)$,
$\tanh\frac{u}{2}=-\tan\frac{\theta}{2}$.]

12·3. Variations and Graphs of Hyperbolic Functions.

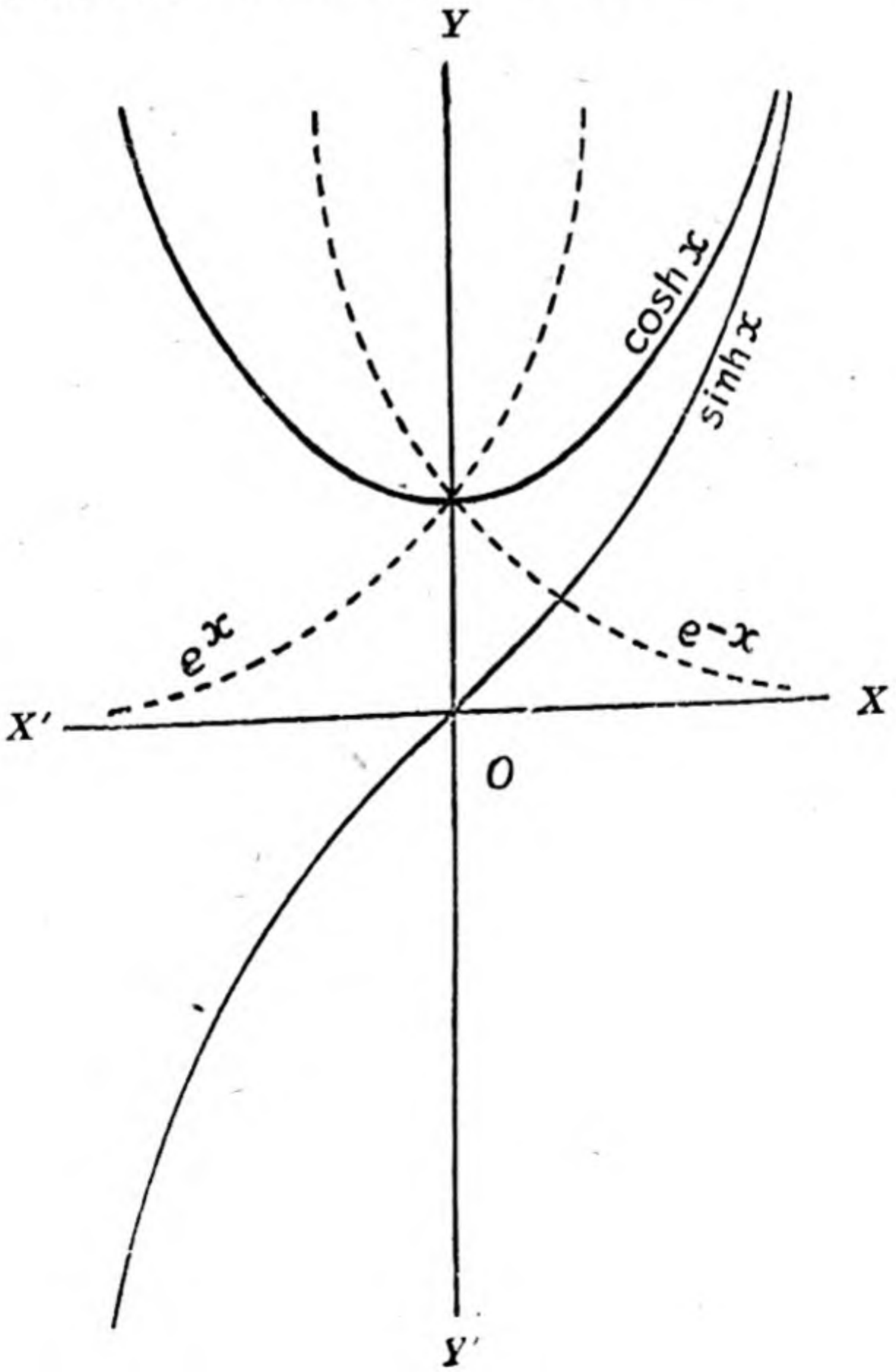
12·31. Cosh x. Cosh x is an even function of x. It

decreases in the interval $(-\infty, 0)$, being large and positive, when x is numerically large but negative. For $x=0$, $\cosh x=1$, which is the minimum value of the function. For $0 \leqslant x$, the function continuously increases with x, and is large for large values of x.

The graph of this function can be constructed from the graphs of e^x, and e^{-x} (shown by dotted lines) by taking for any particular value of x, the ordinate of $y=\cosh x$ to be the mean of the ordinates of $y=e^x$ and e^{-x}.

12·32. Sinh x. Sinh x is an odd function, and vanishes for $x=0$. As x increases in the interval $(-\infty, \infty)$, $\sinh x$ also increases from $-\infty$ to ∞.

The graph of this function could be constructed in the same way as that of $\cosh x$. In this case the ordinate of $y=\sinh x$ will be the mean of the difference of e^x and e^{-x}.

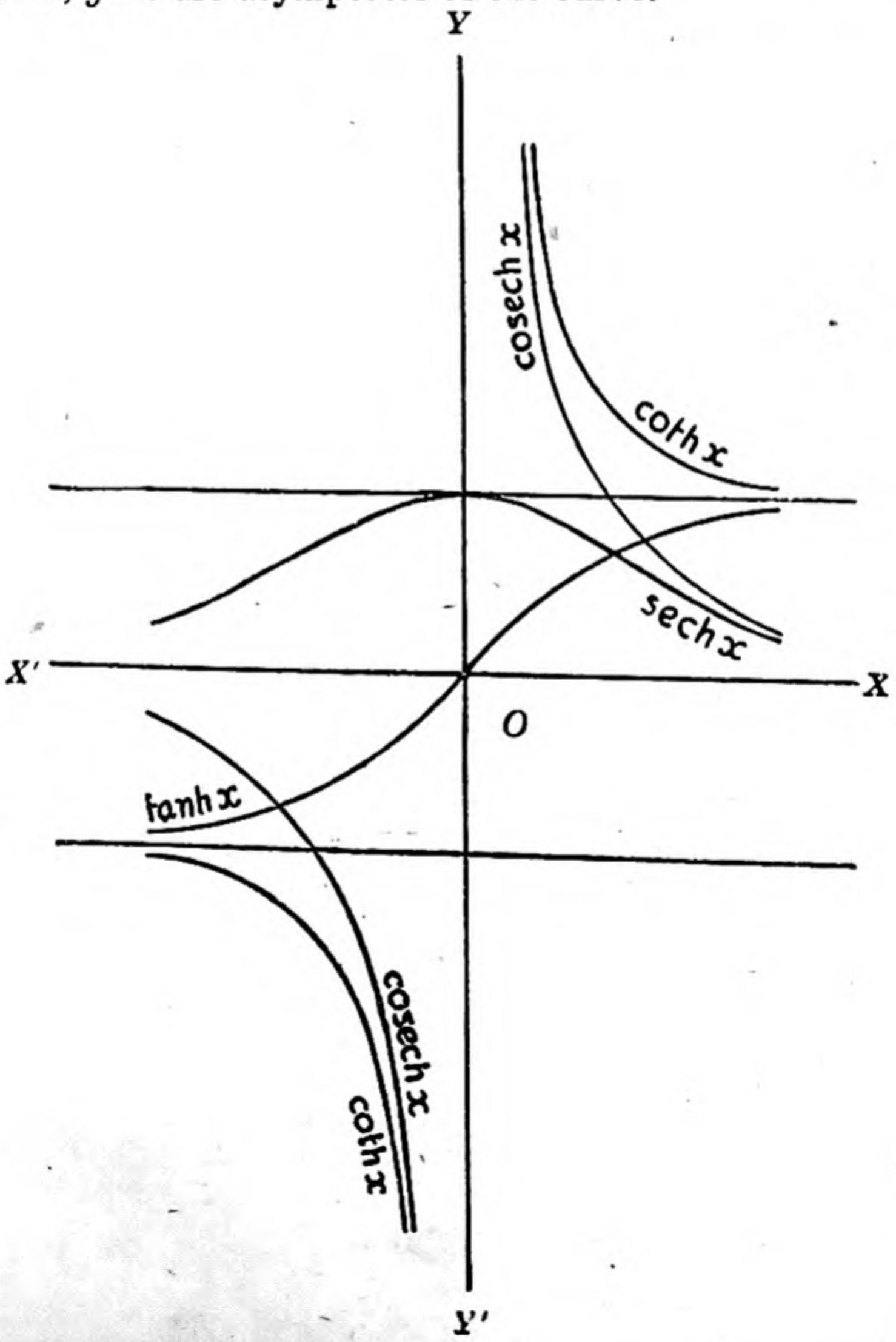


12·33. Tanh x. Tanh x is an odd function and lies between -1 and 1. The lines $y=-1$ and $y=1$ are asymptotes of the curve which it never crosses.

12·34. Sech x. Sech x is an even function, and has the maximum value 1 for $x=0$. It lies above its asymptote $y=0$.

12·35. Coth x. Coth x is an odd function and lies above the line $y=1$ and below the line $y=-1$ which it approaches when $x\to\infty$ and $x\to-\infty$ respectively. For $x=0$, $y\to\infty$, so that $x=0$ is also an asymptote of the curve.

12·36. Cosech x. Cosech x is an odd function. The lines $x=0$, $y=0$ are asymptotes of the curve.



12·4. Inverse Hyperbolic Functions.

(*a*) Let $\cosh x=y$

then $\dfrac{e^x+e^{-x}}{2}=y$, or $e^{2x}-2ye^x+1=0$;

$\therefore\ e^x=y\pm\sqrt{y^2-1}$, or $x=\log(y\pm\sqrt{y^2-1})$,

so that for given $y>1$, there are two real values of x, which are equal in magnitude but opposite in sign.

The positive value is denoted by $\cosh^{-1} y$, so that

$$\cosh^{-1}y=\log(y+\sqrt{y^2-1}). \tag{1}$$

Thus $\cosh^{-1}y$ is a single-valued and continuous function of $y\geqslant 1$.

(*b*) If $\sinh x=y$,

$$\frac{e^x-e^{-x}}{2}=y, \text{ therefore, } e^{2x}-2y\,e^x-1=0.$$

Hence $e^x=y+\sqrt{y^2+1}$.

The negative sign before the radical is rejected, for e^x is never negative. Thus

$$\sinh^{-1} y=x=\log(y+\sqrt{y^2+1}). \tag{2}$$

$\sinh^{-1}y$ is a single-valued continuous function of y for all values of y.

(*c*) If $\tanh x=y$

$$\frac{e^x-e^{-x}}{e^x+e^{-x}}=y, \quad \therefore\ e^{2x}=\frac{y+1}{1-y}.$$

Thus $\tanh^{-1}y=x=\frac{1}{2}\log\dfrac{1+y}{1-y}$, (3)

which is also a single-valued continuous function of y in the interval $-1<y<1$

In like manner, we can show that

$$\coth^{-1} y=\tfrac{1}{2}\log\frac{y+1}{y-1}, \tag{4}$$

$$\operatorname{sech}^{-1}y=\log\frac{1\pm\sqrt{1-y^2}}{y}, \tag{5}$$

$$\operatorname{cosech}^{-1} y=\log\frac{1+\sqrt{1+y^2}}{y}. \tag{6}$$

Expansion of certain functions in series.

The discussion in the following section depends on the following theorem of analyis which is almost intuitive :—

If $f(x)\geqslant 0$, when $a\leqslant x\leqslant b$, then

$$\int_a^b f(x)\,dx\geqslant 0.$$

The value of the integral cannot be zero unless $f(x)$ is always equal to zero.

For a proof of this theorem reference may be made to Hardy's Pure Mathematics, Seventh Edition, Page 321).

12·51. Gregory's series for $\tan^{-1}x$.

We start with the formula

$$\tan^{-1}x=\int_0^x \frac{dt}{1+t^2}, \quad \text{where } -\frac{\pi}{2}<x<\frac{\pi}{2}.$$

Now by division,

$$\frac{1}{1+t^2}=1-t^2+t^4+\ldots+(-1)^{n-1}\ t^{2n-2}+(-1)^n\ \frac{t^{2n}}{1+t^2}.$$

Integrating between the limits 0 to x, we have

$$\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots+(-1)^{n-1}\ \frac{x^{2-1}}{2n-1}+R_n(x),$$

where $$R_n(x)=(-1)^n\int_0^x \frac{t^{2n}}{1+t^2}dt.$$

If $x>0$, $|R_n(x)|<\int_0^x t^{2n}.dt=\frac{x^{2n+1}}{2n+1}$.

If $0<x<1$, $\frac{x^{2n+1}}{2n+1}\longrightarrow 0$, as $n\longrightarrow\infty$. It follows, therefore, that

$$\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots \text{ ad infinitum, } 0\leqslant x<1.$$

If x is $-$ve, and $-1<x<0$, put $x=-y$, so that $0<y<1$,

$$\tan^{-1}\ x=-\tan^{-1}y=-\left(y-\frac{y^3}{3}+\frac{y^5}{5}-\frac{x^7}{7}+\cdots\right)$$
$$=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$$

The series also converges for $x=1$, being an alternating series whose nth term tends to zero (7·73).

Thus $\tan^{-1}x=x-\frac{x^3}{4}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots -1<x\leqslant 1.$

12·52 Expansions of $\sin x$ and $\cos x$. We shall prove that for all real values of x

$$\sin x=x-\frac{x^3}{4\,!}+\frac{x^5}{5\,!}+\cdots=\sum_{r=0}^{\infty}(-1)^r\frac{x^{2r+1}}{(2r+1)\,!} \qquad (1)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{r=0}^{\infty} (-1)^{2r} \frac{x^{2r}}{(2r)!}. \qquad (2)$$

Since $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$, if the above results can be proved true when $x > 0$, they will also remain true when $x < 0$.

Put $$S_{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!},$$

$$C_{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!}.$$

First we shall prove that $S_{2n+1} \gtrless \sin x$, according as n is even or odd ; and $C_{2n} \gtrless \cos x$, according as n is even or odd,

i.e., $S_{4m+1} > \sin x > S_{4m-1}$ $\quad n = 2m,\ n = 2m-1$; (3)

$C_{4m} > \cos x > C_{4m-2}$ $\quad n = 2m,\ n = 2m-1$; (4)

Proof. For all real values of x

$$1 - \cos x \geqslant 0.$$

Assuming $x > 0$, $$\int_0^x (1 - \cos x)\, dx \geqslant 0,$$

for the integrand is positive, and x is increasing during the range of integration. . Thus

$$x - \sin x \geqslant 0.$$

Integrating repeatedly we get

$$\frac{x^2}{2!} - 1 + \cos x > 0, \textit{ i.e.}, \cos x - C_2 > 0, \quad x > 0$$

$$\frac{x^3}{3!} - x + \sin x > 0, \textit{ i.e.}, \sin x - S_3 > 0, \quad x > 0$$

$$\frac{x^4}{4!} - \frac{x^2}{2!} + 1 - \cos x > 0, \textit{ i.e.}, C_4 - \cos x > 0, \quad x > 0$$

$$\frac{x^5}{5!} - \frac{x^3}{3!} + x - \sin x > 0, \textit{ i.e.}, S_5 - \sin x > 0, \quad x > 0$$

Thus we get a series of inequalities of which the general forms are given by (3) and (4). We now prove the general case by induction, assuming one part of the inequalities, (3) or (4), say

$$\cos x - C_{4m-2} > 0. \qquad (5)$$

Integrating this between the range 0 to x

$$\sin x - S_{4m-1} > 0. \qquad (6)$$

Integrating again,

$$C_{4m} - \cos x > 0. \qquad (7)$$

Integrating twice again,

$$S_{4m+1}-\sin x>0, \tag{8}$$

$$\cos x-C_{4m+2}>0. \tag{9}$$

Thus if (5) holds for any value of n, the results (6), (7), (8), (9) also hold. But it holds for $n=2$; hence (3), and (4) hold.

Thus $\cos x=C_{4m-2}+R$, where $R<\dfrac{x^{4n}}{(4n)!}$,

$$=C_{4m}-R' \quad , \text{ where } R'<\frac{x^{4n+2}}{(4n+2)!}.$$

Obviously, $\dfrac{x^{4n}}{(4n)!}$ and $\dfrac{x^{4n+2}}{(4n+2)!}$ both tend to zero, as $n\to\infty$.

This proves (2).

The result of (1) follows similarly from (3).

12·53. The series for log (1+x).

The function $\log x$ is defined by the integral

$$\log x=\int_1^x \frac{du}{u}, \quad x>0.$$

From the definition, therefore,

$$\log (1+x)=\int_1^{1+x} \frac{du}{u}, \quad (1+x>0, \text{ or } x>-1)$$

$$=\int_0^x \frac{dt}{1+t}.$$

Now by division,

$$\frac{1}{1+t}=1-t+t^2-t^3+\ldots\ldots+(-1)^n\frac{t^n}{1+t}.$$

Hence on integrating term by term, it is found if $x>-1$,

$$\log (1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots+(-1)^{n-1}\frac{x^n}{n}+R_n(x),$$

where

$$R_n(x)=(-1)^n\int_0^x \frac{t^n}{1+t}\,dt.$$

If $x>0$,

$$\left|R_n(x)\right|=\int_0^x \frac{t^n}{1+t}\,dt<\int_0^x t^n dt=\frac{x^{n+1}}{n+1}.$$

Hence if $0<x\leqslant 1$,

$$|R_n(x)| \to 0, \text{ as } n\to\infty.$$

If $-1<x<0$, put $x=-y$, so that $y>0$.

$$R_n(x)=(-1)^n\int_0^{-y}\frac{t^n}{1+t}\,dt.$$

Putting $t=-v$,

$$R_n(x)=\int_0^{y}\frac{v^n}{1-v}\,dv.$$

Now since $1>y>0$, in the range of integration $(0, y)$, $1-y\leqslant 1-v$;

$$\therefore\quad R_n(x)<\int_0^{y}\frac{1}{1-y}v^n dv$$

$$=\frac{1}{1-y}\,\frac{y^{n+1}}{n+1}$$

$$\leqslant\frac{1}{1+x}\,\frac{|x|^{n+1}}{n+1}.$$

Since $-1<x<0$, $R_n(x)\to 0$, as $n\to\infty$.

Hence if $-1<x\leqslant 1$,

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\cdots$$

In particular, for $x=1$,

$$\log 2=1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}\cdots\cdots$$

12·6. Convergence of Infinite Series whose terms are Functions of the Complex Variable z=x+i y.

Let $w_1+w_2+w_3+\cdots$ be the series, where w_1, w_2,...are all functions of the complex variable, $z=x+i\,y$. Suppose

$$w_n=u_n+i\,v_n,$$

so that u_n and $i\,v_n$ are the real and inaginary parts of w_n. Denote by W_n, U_n, V_n the sums of the first n terms of Σw_n, Σu_n, Σv_n, then

$$W_n=U_n+i\,V_n.$$

If U_n and V_n both tend to limits U and V respectively, then W_n is said to tend to the limit $U+i\,V$, so that

$$W=\operatorname*{Lt}_{n\to\infty} W_n=U+i\,V.$$

We then say that the series converges and the sum is W, so that Σw_n is said to be convergent if Σu_n and Σv_n converge separately.

Theorem. *A series of complex terms converges, if the series of moduli is convergent.*

For $$|u_n| < (u_n^2+v_n^2)^{\frac{1}{2}}, \quad |v_n| < (u_n^2+v_n^2)^{\frac{1}{2}},$$

and since $\Sigma(u_n^2+v_n^2)^{\frac{1}{2}}$ is given to be convergent, Σu_n and Σv_n therefore converge separately (7·33). Thus Σw_n has a finite sum, as $n\to\infty$, *i.e.*, the series converges.

A series of complex terms is said to be absolutely convergent if the series of its moduli is convergent.

12·7. Exponential Function of a Complex Variable.

When z is real, the infinite series

$$1+\frac{z}{1!}+\frac{z^2}{2!}+\cdots+\frac{z^n}{n!}+\cdots \qquad (1)$$

has been seen to be convergent and its sum has been denoted by the exponential function e^z. If z is complex and $=r(\cos\theta+i\sin\theta)$, no significance has been attached to this infinite series. The series of moduli

$$1+\frac{r}{1!}+\frac{r^2}{2!}+\cdots+\frac{r^n}{n!}+\cdots$$

is convergent for all values of r (page 207, Ex. 11). The series (1) is, therefore, absolutely convergent for all finite values of z.

We define e^z, when z is complex, to be the sum of the infinite series (1).

This is a finite and single-valued function.

A property of absolutely convergent series is that they can be multiplied term by term (7·931). We can then prove in precisely the same manner as in (Art. 12·1) that

$$e^z.\,e^{z'}=e^{z+z'}$$

In particular, $$e^{x+iy}=e^x.\,e^{iy}.$$

12·71. Exponential Values of Circular Functions.

If $z=ix$,

$$e^{iz}=1+\frac{ix}{1!}-\frac{x^2}{2!}-\frac{ix^3}{3!}+\frac{x^4}{4!}+\cdots$$

$$=\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)+i\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)$$

$$=\cos x+i\sin x,$$

the grouping being admissible by (7·912).

Similarly, $e^{-iz}=\cos x-i\sin x$

$$\therefore \quad \cos x=\frac{e^{ix}+e^{-ix}}{2}, \quad \sin x=\frac{e^{ix}-e^{-ix}}{2i}.$$

12·72. Generalized Circular and Hyperbolic Functions. The circular functions of a complex variable z are defined by the equations:—

$$\left.\begin{aligned} &\cos z=\frac{e^{iz}+e^{-iz}}{2}, \quad \sin z=\frac{e^{iz}-e^{-iz}}{2i}, \quad \tan z=\frac{\sin z}{\cos z}, \\ &\sec z=\frac{1}{\cos z}, \quad \operatorname{cosec} z=\frac{1}{\sin z}. \quad \cot z=\frac{\cos z}{\sin z}=\frac{1}{\tan z}. \end{aligned}\right\} \qquad (1)$$

The close analogy between the *definitions* of generalized circular functions, as given above, and the relations *proved* in (12·71) for circular functions of a real variable, show that the above definitions include as particular cases the circular functions of a real variable. All the formulae which hold for circular functions of a real variable remain true for circular functions of a complex variable.

In a similar way we may define the generalized hyperbolic functions by the equations

$$\left.\begin{aligned} &\cosh z=\tfrac{1}{2}(e^{z}+e^{-z}), \ \sinh z=\tfrac{1}{2}(e^{z}-e^{-z}), \\ &\tanh z=\frac{e^{z}-e^{-z}}{e^{z}+e^{-z}}, \quad \coth z=\frac{e^{z}+e^{-z}}{e^{z}-e^{-z}}, \\ &\operatorname{sech} z=\frac{2}{e^{z}+e^{-z}}, \quad \operatorname{cosech} z=\frac{2}{e^{z}+e^{-z}}. \end{aligned}\right\} \qquad (1)$$

The results that have been obtained for hyperbolic functions of a real variable hold good in the general case. We proceed to establish the relations between the circular and hyperbolic functions:—

If z is purely imaginary and equal to ix, where x is real,

$$\left.\begin{aligned} &\sin (ix)=(e^{-x}-e^{x})/2i=i\sinh x, \\ &\cos (ix)=\tfrac{1}{2}(e^{-x}+e^{x})=\cosh x. \\ &\therefore \ \tan (ix)=i\tanh x, \ \sec ix=\operatorname{sech} x, \\ &\operatorname{cosec} (ix)=-i\operatorname{cosech} x, \ \cot (ix)=-i\coth x. \end{aligned}\right\} \qquad (2)$$

$$\left.\begin{aligned} &\sinh (ix)=\tfrac{1}{2}(e^{ix}-e^{-ix})=i\sin x. \\ &\qquad\qquad \therefore \quad \operatorname{cosech} (ix)=-i\operatorname{cosec} x \\ &\cosh (ix)=\tfrac{1}{2}(e^{ix}+e^{-ix})=\cos x \\ &\qquad\qquad \therefore \quad \operatorname{sech} (ix)=\sec x \\ &\tanh (ix)=i\tan x \qquad\qquad \coth (ix)=-i\cot x. \end{aligned}\right\}$$

It will be instructive to get certain formulae from these definitions.

(*a*) $\cos z+i\sin z=e^{z}$, $\cos z-i\sin z=e^{-z}$.

Multiplication gives $\cos^2 z+\sin^2 z=1$.

(*b*) $\sin u \cos v=(e^{iu}-e^{-iu})\ (e^{iv}+e^{-iv})/4i$

$=[(e^{i(u+v)}-e^{-i(u+v)})+(e^{i(u-v)}-e^{-i(u-v)})]/4i.$

$\therefore\ 2\sin u\cos v=\sin(u+v)+\sin(u-v).$

Again, $\cos u \sin v=(e^{iu}+e^{-iu})\ (e^{iv}-e^{-iv})/4i$

$=[(e^{i(u+v)}-e^{-i(u+v)})-(e^{i(u-v)}-e^{-i(u-v)})]/4i,$

or, $2\cos u\sin v=\sin(u+v)-\sin(u-v)$.

Addition and subtraction give

$$\left.\begin{aligned}\sin(u+v)&=\sin u\cos v+\cos u\sin v,\\ \sin(u-v)&=\sin u\cos v-\cos u\sin v.\end{aligned}\right\}\qquad(3)$$

Similarly, $\cos(u\pm v)=\cos u\cos v\mp\sin u\sin v$,

12·721. By application of (2) we deduce from formulae for circular functions, the corresponding results for hyperbolic functions. (See art. 12·211).

Illustration. Putting ix for x in the formula $\sin 2x=\dfrac{2\tan x}{1+\tan^2 x}$,

$\sin 2ix=\dfrac{2\tan ix}{1+\tan^2 ix}$, whence $\sinh 2x=\dfrac{2\tanh x}{1-\tanh^2 x}$.

In (14·14) writing iu, iv for u and v, we get

$\sin i(u+v)=\sin iu\cos iv+\cos iu\sin iv,$

$\therefore\ i\sinh(u+v)=i\sinh u\cosh v+i\cosh u\sinh v,$

or, $\sinh(u+v)=\sinh u\cosh v+\cosh u\sinh v.$

Similarly, $\sinh(u-v)=\sinh u\cosh v-\cosh u\sinh v.$

Making the same substitution in (4)

$\cos i(u\pm v)=\cos iu\cos iv\mp\sin iu\sin iv,$

or $\cosh(u\pm v)=\cosh u\cosh v\pm\sinh u\sinh v.$

From relations

$$\cos z=1-\frac{z^2}{2!}+\frac{z^4}{4!}\cdots\cdots$$

$$\sin z=z-\frac{z^3}{3!}+\frac{z^5}{5!}\cdots\cdots$$

We deduce $\cosh z=1+\dfrac{z^2}{2!}+\dfrac{z^4}{4!}+\cdots\cdots$

$$\sinh z=z+\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots\cdots$$

12·73. Example 1. If $\alpha+i\beta=i^{\alpha+i\beta}$, prove that

$$\alpha^2+\beta^2=e^{-(4n+1)\pi\beta}.\qquad(P.\ U.\ 1941)$$

Since i may be put as $\cos\dfrac{\pi}{2}+i\sin\dfrac{\pi}{2}$, $\alpha+i\beta=\left(\operatorname{cis}\dfrac{\pi}{2}\right)^{\alpha+i\beta}$

$$=\left(\operatorname{cis}\overline{2n\pi+\frac{\pi}{2}}\right)^{\alpha+i\beta}.$$

Putting $(2n+\frac{1}{2})\pi$ or $\frac{4n+1}{2}.\pi=k$, the above becomes

$\alpha+i\beta=\text{cis}\,(k\alpha+ik\beta)$ [By De Moivre's Theorem Case

$=\cos k\alpha\cos i\beta k-\sin k\alpha\sin i\beta k+i\,(\sin k\alpha\cos i\beta k+\cos k\alpha\sin i\beta k)$

$=\cos k\alpha\cosh\beta k-\cos k\alpha\sinh\beta k+i(\sin k\alpha\cosh\beta k-\sin k\alpha\sinh\beta k)$.

Equating the real and imaginary parts,

$\alpha=\cos k\alpha\,(\cosh\beta k-\sinh\beta k)$,

$\beta=\sin k\alpha\,(\cosh\beta k-\sinh\beta k)$.

Squaring and adding, $\alpha^2+\beta^2=(\cosh\beta k-\sinh\beta k)^2$

$=(e^{-\beta k})^2=e^{-2k\beta}=e^{-(4n+1)\pi\beta}$.

Examples XII(c)

Prove the following relations :—

1. $\dfrac{\cosh^2u-1}{\cosh^2u+1}=\tanh^2u$.

2. $\dfrac{\sinh u}{\cosh u+1}=\tanh\dfrac{u}{2}=\dfrac{\cosh u-1}{\sinh u}$.

3. $\dfrac{1+\tanh u}{1-\tanh u}=\cosh 2u+\sinh 2u$.

4. (*a*) $\sinh(\alpha+\beta+\gamma)-\sinh\alpha-\sinh\beta-\sinh\gamma$

$=4\sinh\frac{1}{2}(\beta+\gamma)\sinh\frac{1}{2}(\gamma+\alpha)\sinh\frac{1}{2}(\alpha+\beta)$.

(*b*) $\cosh(\alpha+\beta+\gamma)-\cosh\alpha-\cosh\beta-\cosh\gamma$

$=4\cosh\frac{1}{2}(\beta+\gamma)\cosh\frac{1}{2}(\gamma+\alpha)\cosh\frac{1}{2}(\alpha+\beta)$.

5. If $\cosh u=\sec\theta$, prove that $\sinh u=\tan\theta$, $\tanh u=\sin\theta$.

6. Prove that $\sin 2\alpha+i\sinh 2\beta=2\sin(\alpha+i\beta)\cos(\alpha-i\beta)$.

7. Prove that

$\cos(\alpha+i\beta)+i\sin(\alpha+i\beta)=e^{-\beta}(\cos\alpha+i\sin\beta)$.

8. Express $\sin(x+iy)$ in the form $A+iB$.

[**Solution :** $\sin(x+iy)=\sin x\cos iy+\cos x\sin iy$

$=\sin x\cosh y+i\cos x\sinh y$.

Thus $\boldsymbol{R}\{\sin(x+iy)\}=\sin x\cosh y$,

$\boldsymbol{I}\{\sin(x+iy)\}=\cos x\sinh y$.]

9. Separate $\cos(x+iy)$ into real and imaginary parts.

10. Prove that $\tan(x+iy)=\dfrac{\sin 2x+i\sinh 2y}{\cos 2x+\cosh 2y}$.

[**Solution :** $\tan(x+iy)=\dfrac{\sin(x+iy)}{\cos(x+iy)}$

$=\dfrac{2\sin(x+iy)\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)}$

$$\therefore \tan(x+iy)=\frac{\sin 2x+\sin 2iy}{\cos 2x+\cos 2iy}$$
$$=\frac{\sin 2x+i\sinh 2y}{\cos 2x+\cosh 2y}.\ ]$$

11. Prove that $\cot(x+iy)=\dfrac{\sin 2x-i\sinh 2y}{\cosh 2y-\cos 2x}$.

12. Prove that $\sinh(x+iy)=\sinh x\cos y+i\cosh x\sin y$.

13. Prove that

(*i*) $\sec(x+iy)=2\dfrac{\cos x\cosh y+i\sin x\sinh y}{\cosh 2x+\cosh 2y}$.

(*ii*) $\operatorname{cosec}(x+iy)=\dfrac{2\sin x\cosh y-i\cos x\sinh y}{\cosh 2y-\cos 2x}$.

14. Find the value of $|\sin(x+iy)|$

[Solution: If $\sin(x+iy)=P+iQ$,
then $\sin(x-iy)=P-iQ$.
$\therefore\ P^2+Q^2=|\sin(x+iy)|^2=\sin(x+iy)\sin(x-iy)$
$=\frac{1}{2}[\cos 2iy-\cos 2x]$
$=\frac{1}{2}(\cosh 2y-\cos 2x)=\cosh^2 y-\cos^2 x$;
$\therefore\ |\sin(x+iy)|=\sqrt{\cosh^2 y-\cos^2 x}$.]

15. Prove that $|\cos(x+iy)|=\sqrt{\cos^2 x+\sinh^2 y}$.

16. Show that $|\tan(x+iy)|^2=\dfrac{\cosh 2y-\cos 2x}{\cos 2x+\cosh 2y}$.

17. If $\tan(x+iy)=\sin(u+iv)$, prove that
$$\frac{\sin 2x}{\sinh 2y}=\frac{\tan u}{\tanh v}.$$

[Solution: $\sin u\cos iv+\cos u\sin iv=\dfrac{\sin(x+iy)}{\cos(x+iy)}$,

$\therefore\ \sin u\cosh v+i\cos u\sinh v=\dfrac{2\sin(x+iy)\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)}$.
$$=\frac{\sin 2x+\sin 2iy}{\cos 2x+\cos 2iy}$$
$$=\frac{\sin 2x+i\sinh 2y}{\cos 2x+\cosh 2y}$$

$\therefore\ \sin u\cosh v=\dfrac{\sin 2x}{\cos 2x+\cosh 2y}$,

$\cos u\sinh v=\dfrac{\sinh 2y}{\cos 2x+\cosh 2y}$.

Dividing $\dfrac{\sin u \cosh v}{\cos u \sinh v}=\dfrac{\sin 2x}{\sinh 2y}$,

$\therefore \quad \dfrac{\tan u}{\tanh v}=\dfrac{\sin 2x}{\sinh 2y}$.]

18. Prove that $\cos (x+iy)=-\dfrac{\sin 2x-i \sinh 2y}{\cos 2x-\cosh 2y}$.

19. If $|\cos (u+iv)| =1$, then $\sin^2 u=\sinh^2 v$.

20. If $|\sin (u+iv)| =1$, then $\cos^2 u=\sinh^2 v$.

12·8. Periodicity of the Exponential and Hyperbolic Functions.

If $z=x+iy$, $e^z=e^{x+iy}=e^x (\cos y+i \sin y)$.

Now cos y and sin y are unaltered if y be replaced by $y+2n_\pi$ where n is any integer,

$$\therefore \quad e^z=e^x [\cos (y+2n\pi)+i \sin (y+2n_\pi)]$$
$$=e^{x+i (y+2n\pi)}=e^{z+2in\pi}.$$

Thus e^z is periodic with the period $2i_\pi$.

(*b*) Since $e^{2in\pi}=\cos 2n\pi+i \sin 2n\pi=1$,

$$\therefore \quad e^{z+2in\pi}=e^z,$$
$$e^{-z-2in\pi}=e^{-z}.$$

Hence
$$e^{z+2in\pi}+e^{-z-2in\pi}=e^z+e^{-z},$$
$$e^{z+2in\pi}-e^{-z-2in\pi}=e^z-e^{-z}.$$
$$\therefore \quad \cosh (z+2in\pi)=\cosh z,$$
$$\sinh (z+2in\pi)=\sinh z.$$

Thus cosh z and sinh z are periodic and have the period $(2\pi i)$.

(*c*) $\tanh (z+in_\pi)=\dfrac{E(z+in\pi)-E(-z-in\pi)}{E(z+in\pi)+E(-z-in\pi)}$

$$=\frac{E(z).\ E(in\pi)-E(-z)\ E(-in\pi)}{E(z)\ E(in\pi)+E(-z)E\ (-in\pi)}$$
$$=\frac{E(z)\ [E(i\pi)]^n-E(-z)\ [E(-i\pi)]^n}{E(z)\ [E(i\pi)]^n+E(-z)\ [E(-i\pi)]^n}$$
$$=\frac{E(z)-E(-z)}{E(z)+E(-z)}=\tanh z,$$

for $E (i\pi)=\cos \pi+i \sin \pi=-1$,
$E(-i\pi)=\cos \pi-i \sin \pi=-1$.

Thus tanh z has the period $i\pi$.

Examples XII (*d*)

1. Prove that $\cos^2\alpha\cosh^2\beta+\sin^2\alpha\sinh^2\beta=\cosh^2\beta-\sin^2\alpha$.

2. What values must be given to x to express the following as simple hyperbolic functions ?

(*i*) $\sqrt{x^2+a^2}$, (*ii*) $\dfrac{1}{\sqrt{x^2-a^2}}$, (*iii*) $\sqrt{x^2+2x+3}$.

(*iv*) $\dfrac{1}{\sqrt{2x^2+3x+4}}$.

3. If $p=a+ib$, $q=a-ib$ where a and b are real, show that pe^p+qe^q is real.

4. Prove that $\sinh(u-v)+\sinh(u+v)+2\sinh u$

$$=2\sinh u(\cosh v+1)=4\sinh u\cosh^2\frac{v}{2}.$$

5. If $\cos\alpha\cosh\beta=\cos\varphi$, $\sin\alpha\sinh\beta=\sin\varphi$, prove that

$$\sin\varphi=\pm\sin^2\alpha=\pm\sinh^2\beta.$$

6. If $\sin(u+iv)=x+iy$, prove that the point (x, y) lies (*i*) on the ellipse $\dfrac{x^2}{\cosh^2 v}+\dfrac{y^2}{\sinh^2 v}=1$, when u is the parameter, and (*ii*) on the hyperbola $\dfrac{x^2}{\sin^2 u}-\dfrac{y^2}{\cos^2 u}=1$, when v is the parameter.

7. If $x+iy=\cos(u+iv)$, prove that

$$(1+x)^2+y^2=(\cosh v+\cos u)^2,$$
$$(1-x)^2+y^2=(\cosh v-\cos u)^2,$$ (*London*)

8. If $\sin(\theta+i\varphi)=\cos\alpha+i\sin\alpha$, then $\cos^2\theta=\pm\sin\alpha$.

9. If $\tan x=\tanh u\tan\alpha$, $\tan y=\tanh u\cot\alpha$, prove that

$$\tan(x+y)=\sinh 2u\operatorname{cosec} 2\alpha.$$

10. If $\tan(x+iy)=u+iv$, prove that

$$\tan 2x=\frac{2u}{1-u^2-v^2} \text{ and } \frac{1-u^2-v^2}{1+u^2+v^2}=\frac{\cos 2x}{\cosh 2y}.$$

[$\tan 2x=\tan(\overline{x+iy}+\overline{x-iy})$ and $\tan(x-iy)=u-iv$.]

11. If $\tan(u+iv)=x+iy$, prove that

$x^2+y^2+2x\cot 2u=1$ and $x^2+y^2-2y\coth 2v+1=0$.

[Solution : Since $\tan(u+iv)=x+iy$, $\tan(u-iv)=x-iy$,

$$\tan 2u=\tan(\overline{u+iv}+\overline{u-iv})=\frac{\tan(u+iv)+\tan(u-iv)}{1-\tan(u+iv)\tan(u-iv)}$$

$$=\frac{2x}{1-(x^2+y^2)},$$

whence $x^2+y^2+2x\cot 2u-1=0$.

Again, $\tan 2\,iv=\tan\overline{[u+iv}-(u-iv)]$

$$=\frac{\tan(u+iv)-\tan(u-iv)}{1+\tan(u+iv)\tan(u-iv)}=\frac{2iy}{1+x^2+y^2}.$$

$\therefore$ $\tanh\ 2v=\dfrac{2y}{x^2+y^2+1}$. Hence etc.]

12. If $x+iy=c\cosh(\theta+i\varphi)$, prove that
$x^2\operatorname{sech}^2\theta+y^2\operatorname{cosech}^2\theta=c^2$, $x^2\sec^2\varphi-y^2\operatorname{cosec}^2\varphi=c^2$.
(London)

13. Separate the real and imaginary parts of $\tan^{-1}(x+iy)$. *(P. U. 1941)*

[Sol. Put $\tan^{-1}(x+iy)=u+iv$,
and $\tan^{-1}(x-iy)=u-iv$.

$\therefore$ $2u=\tan^{-1}\dfrac{2x}{1-x^2-y^2}$ etc.]

14. If $\tanh(u+iv)=x+iy$, where u, v are real, find x and y in terms of u and v. Find the values of u and v, when $x=1$, $y=1$.

[Solution : Since $x+iy=\tanh(u+iv)$, $x-iy=\tanh(u-iv)$.

$\therefore$ $2x=\tanh(u+iv)+\tanh(u-iv)$

$$=\frac{\sinh(u+iv)\cosh(u-iv)+\cosh(u+iv)\sinh(u-iv)}{\cosh(u+iv)\cosh(u-iv)}.$$

$\therefore$ $x=\dfrac{\sinh 2u}{\cosh 2u+\cosh 2\,iv}=\dfrac{\sinh 2u}{\cosh 2u+\cos 2v}$.

Similarly $2\,iy=\tanh(u+iv)-\tanh(u-iv)$

$$=\frac{\sinh(u+iv)\cosh(u-iv)-\sinh(u-iv)\cosh(u+iv)}{\cosh(u+iv)\cosh(u-iv)},$$

or $iy=\dfrac{\sinh 2iv}{\cosh 2u+\cosh 2\,iv}$.

$\therefore$ $y=\dfrac{\sin 2v}{\cosh 2u+\cos 2v}$.

Thus $\tanh(u+iv)$ is separated into real and imaginary parts.

If $x=y=1$, $\sinh 2u=\sin 2v$.

Also $\cosh 2u+\cos 2v=\sinh 2u=\sin 2v$;

$\therefore$ $(\cosh 2u-\sinh 2u)^2=\cos^2 2v=1-\sin^2 2v=1-\sinh^2 2u$.

or $\cosh^2 2u+\sinh^2 2u-2\cosh 2u\sinh 2u=1-\sinh^2 2u$.

$\therefore$ $\sinh 2u\,(3\sinh 2u-2\cosh 2u)=0$.

$\therefore$ $\tanh 2u=\frac{2}{3}$, $\quad\because\ \sinh 2u\neq 0$, for then $x=0$.

$\therefore$ $\dfrac{e^{2u}-e^{-2u}}{e^{2u}+e^{-2u}}=\frac{2}{3}$. Hence $e^{4u}=5$, or $u=\frac{1}{4}\log 5$.

Again, $(\cos 2v-\sin 2v)^2=\cosh^2 2u=1+\sinh^2 2u=1+\sin^2 2v$,

i.e., $1-2\cos 2v\sin 2v=1+\sin^2 2v$.

$\therefore$ $\sin 2v\,(\sin 2v+2\cos 2v)=0$. $\quad\therefore$ $\tan 2v=-2$.

Thus $v=\frac{1}{2}(n\pi-\tan^{-1}2)$.]

15. If $\sin(x+iy)=\rho(\cos\theta+i\sin\theta)$, prove that $\rho^2=\frac{1}{2}(\cosh 2y-\cos 2x)$ and $\tan\theta=\tanh y\cot x$.

16. If $\tan(u+iv)=\tan x+i\sec x$, prove that

$$e^{2v}=\pm\cot\frac{x}{2} \text{ and } 2u=n_\pi+\frac{\pi}{2}+x.$$

[Solution : $\tan 2u=\tan(u+iv+u-iv)$

$$=\frac{2\tan x}{1-(\tan^2 x+\sec^2 x)}=-\cot x=\tan\left(x+\frac{\pi}{2}\right).$$

$$\therefore\quad 2u=n_\pi+\frac{\pi}{2}+x.$$

Again, $\tan 2iv=\tan(\overline{u+iv}-\overline{u-iv})=\dfrac{2i\sec x}{1+\tan^2 x+\sec^2 x}=i\cos x$,

or $\tanh 2v=\cos x$.

$$\therefore\quad \frac{e^{2v}-e^{2v}}{e^{2v}+e^{-2v}}=\frac{\cos x}{1}, \text{ or } e^{4v}=\frac{1+\cos x}{1-\cos x}=\cot^2\frac{x}{2}.$$

$$\therefore\quad e^{2v}=\pm\cot\frac{x}{2}.]$$

17. If $\cos(a+ib)\cosh(x+iy)=1$, where a, b, x, y are real, prove that, in general, $\tan a\tanh b=\tanh x\tan y$. (*London*)

18. If $\tan(\alpha+i\beta)=i$, α and β being real, prove that α is indeterminate and β is infinite.

19. If $\cos(\theta+i\varphi)\cos(\alpha+i\beta)=1$, prove that $\tanh^2\varphi\cosh^2\beta=\sin^2\alpha$, and $\tanh^2\beta\cosh^2\varphi=\sin^2\theta$.

20. If $\tan y=\tan\alpha\tanh\beta$, $\tan z=\cot\alpha\tanh\beta$, prove that $\tan(y+z)=\sinh 2\beta\operatorname{cosec} 2\alpha$.

12·8. *Separate* $\tan^{-1}(\cos\theta+i\sin\theta)$ *into real and imaginary parts.*

Let $\tan^{-1}(\cos\theta+i\sin\theta)=x+iy$;

then, $$\cos\theta+i\sin\theta=\tan(x+iy)=\frac{\sin 2x+i\sinh 2y}{\cos 2x+\cosh 2y}.$$

Thus $$\cos\theta=\frac{\sin 2x}{\cos 2x+\cosh 2y} \quad \ldots \quad \ldots (1),$$

and $$\sin\theta=\frac{\sinh 2y}{\cos 2x+\cosh 2y} \quad \ldots \quad \ldots (2).$$

Squaring and adding, $(\cos 2x+\cosh 2y)^2=\sin^2 2x+\sinh^2 2y$,

or $\cos^2 2x-\sin^2 2x+\cosh^2 2y-\sinh^2 2y+2\cos 2x\cosh 2y=0$,

or $\cos^2 2x+1-\sin^2 2x+2\cos 2x\cosh 2y=0$,

or $2\cos x(\cosh 2y+\cos 2x)=0$.

$\therefore\quad \cos 2x=0, \quad$ or $\quad \cosh 2y=-\cos 2x$.

As x and y are both real, the second possibility is ruled out, and $\cos 2x=0$, which gives

$$2x=n\pi+\frac{\pi}{2}, \quad \text{or} \quad x=\frac{1}{2}n\pi+\frac{\pi}{4}. \quad \ldots \quad \ldots \quad (3)$$

From (2), $\sin\theta=\dfrac{\sinh 2y}{\cosh 2y}=\tanh 2y.$

$$\therefore \quad 2y=\tanh^{-1}\sin\theta=\frac{1}{2}\log\frac{1+\sin\theta}{1-\sin\theta}=\log\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}$$

$$=\log\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}=\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right).$$

$$\therefore \quad y=\frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Thus (3) and (4) give the real and imaginary parts, respectively.

Examples XII (*e*)

1. If $\tan(u+iv)=(\cos x+i\sin x)$, prove that

$$u=\frac{n\pi}{2}+\frac{\pi}{4}, \text{ and } v=\tfrac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right).$$

2. If $\cosh(x+iy)=\cot(u+iv)$, prove that

$$\sinh 2v\,\operatorname{cosec} 2u+\tanh x\tan y=0,$$

and
$$\coth 2v=-\frac{\cosh 2x+\cos 2y+2}{4\sinh x\sin y}.$$

3. If $\cosh(u+iv)=\tan(x+iy)$, prove that
$\cosh 2u+\cos 2v=2(\cosh 2y-\cos 2x)/(\cosh 2y+\cos 2x)$.

4. Prove that

$$\tan^{-1}(e^{i\theta})=\frac{1}{2}n\pi+\frac{\pi}{4}-\frac{1}{2}i\log\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right). \qquad (P.\ U.\ 1945)$$

$$\left[\left\{\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\}^{-1}=\left(\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right)^{-1}=\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}\right.$$

$$\left.=\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right].$$

CHAPTER XIII

SUMMATION OF ELEMENTARY TRIGONOMETRICAL SERIES

13. The methods employed for summation of trigonometrical series will, in general, be similar to those used for summing up series in Algebra. In fact, most of the trigonometrical series are summed up by reducing the series to (*i*) a Geometrical Progression, (*ii*) a Binomial series, (*iii*) an Exponential series, or (*iv*) a Logarithmic series. Quite often the *method of differences* will be found useful.

13·1. To find the sum of a series of sines or cosines of angles in Arithmetical Progression.

13·11. Sum to n terms the series
$\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\ldots+\sin\{\alpha+(n-1)\beta\}$.

Multiplying each term by $2\sin\frac{\beta}{2}$, we have

$$2\sin\alpha\sin\frac{\beta}{2}=\cos\left(\alpha-\frac{\beta}{2}\right)-\cos\left(\alpha+\frac{\beta}{2}\right),$$

$$2\sin(\alpha+\beta)\sin\frac{\beta}{2}=\cos\left(\alpha+\frac{\beta}{2}\right)-\cos\left(\alpha+\frac{3\beta}{2}\right),$$

$$2\sin(\alpha+2\beta)\sin\frac{\beta}{2}=\cos\left(\alpha+\frac{3\beta}{2}\right)-\cos\left(\alpha+\frac{5\beta}{2}\right),$$

.....

$$2\sin(\alpha+\overline{n-1}\beta)\sin\frac{\beta}{2}=\cos\left(\alpha+\overline{2n-3}\frac{\beta}{2}\right)-\cos\left(\alpha+\overline{2n-1}\frac{\beta}{2}\right).$$

If S_n denotes the required sum, we have by addition

$$2\sin\frac{\beta}{2}S_n=2\sin\frac{\beta}{2}\{\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\ldots$$
$$+\sin(\alpha+\overline{n-1}\beta)\}$$
$$=\cos\left(\alpha-\frac{\beta}{2}\right)-\cos\left(\alpha+\overline{2n-1}\frac{\beta}{2}\right) \quad [\because \text{ other terms cancel in pairs}]$$
$$=2\sin\left(\alpha+n\overline{-1}\frac{\beta}{2}\right)\sin\frac{n\beta}{2}.$$

$$\therefore\ S_n=\sin\left(\alpha+\overline{n-1}\frac{\beta}{2}\right)\sin\frac{n\beta}{2}\Big/\sin\frac{\beta}{2} \qquad \ldots(1)$$

13·12. Sum to n terms the series.

$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\cdots+\cos(\alpha+\overline{n-1}\beta)$

We follow the same method of summation.

Let $C_n=\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\cdots+\cos(\alpha+\overline{n-1}\beta)$.

$\therefore$ Multiplying both sides by $2\sin\frac{\beta}{2}$,

$$2\sin\frac{\beta}{2}\cdot C_n=2\sin\frac{\beta}{2}\cos\alpha+2\sin\frac{\beta}{2}\cos(\alpha+\beta)$$

$$+2\sin\frac{\beta}{2}\cos(\alpha+2\beta)+\cdots+2\sin\frac{\beta}{2}\cos(\alpha+\overline{n-1}\beta)$$

$$=\left\{\sin\left(\alpha+\frac{\beta}{2}\right)-\sin\left(\alpha-\frac{\beta}{2}\right)\right\}+\left\{\sin\left(\alpha+\frac{3\beta}{2}\right)-\sin\left(\alpha+\frac{\beta}{2}\right)\right\}$$

$$+\cdots+\left\{\sin\left(\alpha+\frac{2n-1}{2}\beta\right)-\sin\left(\alpha+\frac{2n-3}{2}\beta\right)\right\}$$

$$=\sin\left(\alpha+\frac{2n-1}{2}\beta\right)-\sin\left(\alpha-\frac{\beta}{2}\right)$$

$$=2\cos\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}.$$

$$\therefore C_n=\cos\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}\Big/\sin\frac{\beta}{2}. \qquad \ldots(2)$$

Note 1. The sums are respectively, $\sin\frac{n\beta}{2}\Big/\sin\frac{\beta}{2}$ times the sine of the average angle in (1), and the cosine of the average angle in (2).

Note 2. The result (2) can be obtained from the result (1) by changing α into $\alpha+\frac{\pi}{2}$, which amounts to changing the sine into cosine, wherever α occurs.

Note 3. An easy way to test the accuracy of the formulae, when quoted from memory, would be to see that they hold when $n=1$, or $n=2$.

13·13. Sum to n terms the series :—

$\cosh u+\cosh(u+v)+\cosh(u+2v)+\cdots$

The *nth* term is $\cosh(u+\overline{n-1}\,v)$.

Multiply each term as in 13·12, by $2\sinh\frac{v}{2}$.

$$\text{Then}\quad 2\cosh u\sinh\frac{v}{2}=\sinh\left(u+\frac{v}{2}\right)-\sinh\left(u-\frac{v}{2}\right),$$

$$2\cosh(u+v)\sinh\frac{v}{2}=\sinh\left(u+\frac{3v}{2}\right)-\sinh\left(u+\frac{v}{2}\right),$$

$$2\cosh(u+2v)\sinh\frac{v}{2}=\sin\left(u+\frac{5v}{2}\right)-\sinh\left(u+\frac{3v}{2}\right),$$

.....

$$2\cosh(u+\overline{n-1}v)\sinh\frac{v}{2}$$

$$=\sin\left(u+\frac{2n-1}{2}v\right)-\sinh\left(u+\frac{2n-3}{2}v\right).$$

By addition and simplification, the sum of the required series is found to be

$$\frac{\sinh\frac{nv}{2}}{\sinh\frac{v}{2}}\cosh\left(u+\frac{n-1}{2}v\right) \qquad \ldots(3)$$

13·14. The method used in the above example is the method of differences. It would be instructive to express the terms in exponential form and reduce the series to a geometrical progression. We illustrate this method by applying it to the following series, though the method of 13·13 is equally applicable.

Sum to *n* terms the series

$$\sinh u+\sinh 2u+\sinh 3u+\ldots+\sinh nu.$$

If S_n denotes the required sum,

$$S_n=\tfrac{1}{2}\left(e^u+e^{2u}+e^{3u}+\ldots\text{to } n \text{ terms}\right)-\tfrac{1}{2}\left(e^{-u}+e^{-2u}+e^{-3u}+\ldots\text{to } n \text{ terms}\right)$$

$$=\tfrac{1}{2}\frac{e^u(1-e^{nu})}{1-e^u}-\tfrac{1}{2}e^{-u}\frac{(1-e^{-nu})}{1-e^{-u}}$$

$$=\tfrac{1}{2}\frac{e^{u/2}(1-e^{nu})}{e^{-u/2}-e^{u/2}}-\tfrac{1}{2}e^{-u/2}\frac{(1-e^{-nu})}{e^{+u/2}-e^{-u/2}}$$

$$=\tfrac{1}{2}\left\{\frac{-(e^{u/2}+e^{-u/2})+e^{(n+1/2)u}+e^{-(n+1/2)u}}{e^{u/2}-e^{-u/2}}\right\}$$

$$=\tfrac{1}{2}\left\{\frac{\cosh\left(n+\frac{1}{2}\right)u-\cosh\frac{u}{2}}{\sinh\frac{u}{2}}\right\}=\frac{\sinh\frac{nu}{2}}{\sinh\frac{u}{2}}\sinh\frac{n+1}{2}v$$

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Examples XIII (a)

Sum to n terms the following series :—

1. $\sin x+\sin 2x+\sin 3x+\cdots\cdots$
2. $\cos x+\cos 2x+\cos 3x+\cdots\cdots$
3. $\sin x+\sin 3x+\sin 5x+\cdots\cdots$
4. $\cos x+\cos 3x+\cos 5x+\cdots\cdots$
5. $\cos\frac{\pi}{2n}+\cos\frac{3\pi}{2n}+\cos\frac{5\pi}{2n}+\cdots\cdots$
6. $\sin x+\sin(x-y)+\sin(x-2y)+\cdots\cdots$
7. $\sin x-\sin 2x+\sin 3x-\sin 4x+\cdots\cdots$

[Hint. Put $\pi-x$ for x in (1)].

8. $\cos x+\sin 2x-\cos 3x-\sin 4x+\cos 5x+\sin 6x+\cdots\cdots$

[Hint. Put $\frac{\pi}{2}-x$ for x in (1).]

9. $\cos x-\sin 2x-\cos 3x+\sin 4x+\cdots\cdots$

[Put $\frac{\pi}{2}+x$ for x in (1)].

10. $\sin x-\cos 2x-\sin 3x+\cos 4x+\sin 5x-\cos 6x-\cdots$
11. $\cos x-\cos(x+y)+\cos(x+2y)-\cos(x+3y)+\cdots$

[Hint. Take $\alpha=x$, $\beta=y+\pi$ in 13·12].

12. $\sin x\cos x+\sin 2x\cos 2x+\sin 3x\cos 3x+\cdots$
13. $\sin^2 x+\sin^2 2x+\sin^2 3x+\cdots$
14. $\cos^2 x+\cos^2 2x+\cos^2 4x+\cdots$
15. $\sin^3 x+\sin^3 2x+\sin^3 3x+\cdots$

[Hint. $\sin^3 x=\frac{3\sin x-\sin 3x}{4}$.]

16. $\cos^3 x+\cos^3 2x+\cos^3 3x+\cdots$
17. $\cosh^2 u+\cosh^2 3u+\cosh^2 5u+\cdots$
18. Sum $\cos\frac{2p\pi}{2n+1}+\cos\frac{4p\pi}{2n+1}+\cos\frac{6p\pi}{2n+1}+\cdots+\cos\frac{2np\pi}{2n+1}$,

where p and n are integers.

What will be the sum if (*i*) p is a multiple of $2n+1$, (*ii*) p is not a multiple of $2n+1$?

19. Show that all the solutions of the equation

$$\cos x+\cos 3x+\cos 5x+\ldots+\cos(2n-1)x=0$$

may be written in the form

$$x=\frac{(2s+1)\pi}{2r}.$$

13·2. Method of Differences. This artifice of putting a term in the *form of a difference* is important, but it is not always easy to do so. The student should have sufficient practice and also remember the forms occurring frequently. If the mode of resolution cannot be discovered even by the trial method explained in the following examples, the result, if known, will be a guide to its discovery.

13·21. Sum the series
$$\operatorname{cosec}\alpha+\operatorname{cosec}2\alpha+\operatorname{cosec}4\alpha+\ldots\cos 2^{n-1}\alpha.$$

Let us try to put $\operatorname{cosec}\alpha$ into the form of a difference of two other terms.

$$\operatorname{cosec}\alpha=\frac{1}{\sin\alpha}=\frac{1}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}$$

$$=\frac{}{\sin\frac{\alpha}{2}}-\frac{}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}=\frac{}{\sin\frac{\alpha}{2}}-\frac{}{\sin\alpha},$$

and by trial $\operatorname{cosec}\alpha=\dfrac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}-\dfrac{\cos\alpha}{\sin\alpha}$.

Hence $$\operatorname{cosec}\alpha=\cot\frac{\alpha}{2}-\cot\alpha.$$

Similarly
$$\operatorname{cosec}2\alpha=\cot\alpha-\cot 2\alpha,$$
$$\operatorname{cosec}4\alpha=\cot 2\alpha-\cot 4\alpha,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\operatorname{cosec}2^{n-1}\alpha=\cot 2^{n-2}-\cot 2^{n-1}\alpha.$$

By addition, the sum $=\cot\frac{\alpha}{2}-\cot 2^{n-1}\alpha.$

Note. If S_n be known, putting $n=1$, we get

$$\operatorname{cosec}\alpha=\cot\frac{\alpha}{2}-\cot 2^{0}.\alpha=\cot\frac{\alpha}{2}-\cot\alpha.$$

13·22. Sum to n terms the series
$$\tan\alpha+\frac{1}{2}\tan\frac{\alpha}{2}+\frac{1}{2^2}\tan\frac{\alpha}{2^2}+\ldots$$

It be sometimes quite difficult to split up a term. Trials like the one in the above example may be carried on.

Here if we put $\tan\alpha=\dfrac{\sin\alpha}{\cos\alpha}$, it cannot serve the purpose, for $\cos\alpha$ or $\cos 2\alpha$ cannot be factorized.

Try $\frac{\cos\alpha}{\sin\alpha}-\frac{}{\sin\alpha\cos\alpha}=\frac{\cos\alpha}{\sin\alpha}-\frac{}{\frac{1}{2}\sin 2\alpha}$, and we note that

$$\frac{\cos\alpha}{\sin\alpha}-\frac{2\cos 2\alpha}{\sin 2\alpha}=\frac{\sin\alpha-\sin\alpha\cos 2\alpha}{\sin\alpha\sin 2\alpha}=\frac{\sin\alpha\, 2\sin^2\alpha}{2\sin^2\alpha\cos\alpha}=\tan\alpha.$$

Hence $\tan\alpha=\cot\alpha-2\cot 2\alpha,$

$$\tfrac{1}{2}\tan\frac{\alpha}{2}=\tfrac{1}{2}\cot\frac{\alpha}{2}-\cot\alpha,$$

$$\frac{1}{2^2}\tan\frac{\alpha}{2^2}=\frac{1}{2^2}\cot\frac{\alpha}{2^2}-\tfrac{1}{2}\cot\frac{\alpha}{2},$$

.....

$$\frac{1}{2^{n-2}}\tan\frac{\alpha}{2^{n-1}}=\frac{1}{2^{n-1}}\cot\frac{\alpha}{2^{n-1}}-\frac{1}{2^{n-1}}\cot\frac{\alpha}{2^{n-1}}.$$

By addition

$$\tan\alpha+\frac{1}{2}\tan\frac{\alpha}{2}+\frac{1}{2^2}\tan\frac{\alpha}{2^2}+\ldots\text{to } n \text{ terms}$$

$$=\frac{1}{2^{n-1}}\cot\frac{\alpha}{2^{n-1}}-2\cot 2\alpha.$$

13·3. From sums of known series, the sums of other series can be obtained by *differentiation*, or *integration*.

Thus differentiating the result of art. 13·11 with respect to α, the result of art. 13·12 is obtained, and vice-versa. Similarly 13·13 can be differentiated or integrated to obtain the sum of a series of hyperbolic sines in A.P.

Integrating 13·13 with respect to u, $\sinh u+\sinh(u+v)+\ldots$ to

$$n \text{ terms}=\frac{\sinh\frac{nv}{2}}{\sinh\frac{v}{2}}\cdot\sinh\left(u+\frac{n-1}{2}v\right)+C.$$

Putting $u=0$ and $v=0$, it is easily seen that $C=0$.

Differentiating 13·22, we get

$$\sec^2\alpha+\frac{1}{2^2}\sec^2\frac{\alpha}{2}+\frac{1}{2^4}\sec^2\frac{\alpha}{2^2}+\ldots+\frac{1}{2^{2n-2}}\sec^2\frac{\alpha}{2^{n-1}}$$

$$=-\frac{1}{2^{2n-2}}\operatorname{cosec}^2\frac{\alpha}{2^{n-1}}+4\operatorname{cosec}^2 2\alpha \qquad \ldots (6)$$

It may be noted that Integration of 13·22 gives

$$\log\cos\alpha+\log\cos\frac{\alpha}{2^2}+\log\cos\frac{\alpha}{2^2}+\ldots\text{to } n \text{ terms}$$

$$=-\log\sin\frac{\alpha}{2^{n-1}}+\log\sin 2\alpha+\text{constant},$$

or $$\cos\alpha\cos\frac{\alpha}{2}\cos\frac{\alpha}{2^2}\ldots\text{to } n \text{ factors}=K.\frac{\sin 2\alpha}{\sin\frac{\alpha}{2^{n-1}}},$$

which is only the result of factorising $\sin 2a$ by a continued process.

Conversely, from the product of $\sin 2a$, the result 13·22 could be obtained by differentiation.

Examples XIII (*b*)

Sum to n terms :—

1. $\sin x \sin 2x+\sin 2x \sin 3x+\sin 3x \sin 4x+\cdots$
2. $\sin x \sin 3x+\sin 2x \sin 6x+\sin 4x \sin 12x+\cdots$
3. $\sinh u \cosh 2u+\sinh 3u \cosh 4u+\sinh 5u \cosh 6u+\cdots$

Sum each of the following series to n terms by the *method of differences* :—

4. $\operatorname{cosec} a+\operatorname{cosec} \frac{a}{2}+\operatorname{cosec} \frac{a}{2^2}+\cdots$
5. $\tan a+2 \tan 2a+4 \tan 4a+8 \tan 8a+\cdots$
6. $\tan a \tan 2a+\tan 2a \tan 3a+\tan 3a \tan 4a+\cdots$

[Hint. $\tan a \tan 2a$ can be got from $\tan (2a-a)$.]

7. $\tan a \sec 2a+\tan 2a \sec 2^2 a+\tan 2^2 a \sec 2^3 a+\cdots$

[Hint. $\tan 2a-\tan a=\tan a \sec 2a$].

8. $\dfrac{\sin 2a}{\sin a \sin 3a}+\dfrac{\sin 2a}{\sin 3a \sin 5a}+\dfrac{\sin 2a}{\sin 5a \sin 7a}+\cdots$
9. $\dfrac{1}{\cos a \cos 3a}+\dfrac{1}{\cos 3a \cos 5a}+\dfrac{1}{\cos 5a \cos 7a}+\cdots$
10. $\sec a \sec 2a+\sec 2a \sec 3a+\sec 3a \sec 4a+\cdots$
11. $\dfrac{\sin 2a}{\cos a \cos 3a}+\dfrac{\sin 4a}{\cos 3a \cos 5a}+\dfrac{\sin 6a}{\cos 5a \cos 7a}+\cdots$

[Hint. $\dfrac{1}{\cos 3a}-\dfrac{1}{\cos 5a}$ will lead to the second term.]

12. $\dfrac{\cos 2a}{\sin a \sin 3a}+\dfrac{\cos 4a}{\sin 3a \sin 5a}+\dfrac{\cos 6a}{\sin 5a \sin 7a}+\cdots$
13. $\sin^3 a+\frac{1}{3} \sin^3 3a+\frac{1}{3^2} \sin^3 3^2 a+\cdots$
14. $\cos a \operatorname{cosec} 3a+\cos 3a \operatorname{cosec} 3^2 a+\cos 3^2 a \operatorname{cosec} 3^3 a+\cdots$
15. $\cot \theta \cot 2\theta+\cot 2\theta \cot 3\theta+\cot 3\theta \cot 4\theta+\cdots$

Prove the formula, $p>q$

$$\operatorname{cosec} qx \operatorname{cosec} px=\operatorname{cosec} (p-q)\,[\cot qx-\cot px].$$

Deduce

16. $\operatorname{cosec} x \operatorname{cosec} 2x+\operatorname{cosec} 2n \operatorname{cosec} 3x+\cdots$ to n terms $=\operatorname{cosec} x\,[\cot x-\cot(n+1)x]$.
17. $\operatorname{cosec} x \operatorname{cosec} 3x+\operatorname{cosec} 3x \operatorname{cosec} 5x+\cdots$ to n terms $=\operatorname{cosec} 2x\,[\cot x-\cot (2n+1)x]$

18. $\operatorname{cosec}\left(\theta+\frac{\pi}{4n}\right)\operatorname{cosec}\left(\theta+\frac{3\pi}{4n}\right)+\operatorname{cosec}\left(\theta+\frac{3\pi}{4n}\right)\operatorname{cosec}\left(\theta+\frac{5\pi}{4n}\right)$
$+\cdots$to n terms.
$$=2\operatorname{cosec}\frac{\pi}{2n}\operatorname{cosec}\left(\frac{\pi}{2n}+2\theta\right).$$

19. Sum the series
$\operatorname{cosec}\alpha\operatorname{cosec}(\alpha+\beta)+\operatorname{cosec}(\alpha+\beta)\operatorname{cosec}(\alpha+2\beta)$
$+\operatorname{cosec}(\alpha+2\beta)\operatorname{cosec}(\alpha+3\beta)+\cdots$to n terms.

20. Sum the series
$\tan 2\theta\tan^2\theta+\frac{1}{2}\tan 4\theta\tan^2 2\theta+\frac{1}{4}\tan 8\theta\tan^2 4\theta+\cdots$
to n terms. *(P. U. 1945)*

[*Cf.* Ex. 6 above, and establish $\tan^2\theta\tan 2\theta=\tan 2\theta-2\tan\theta$]

21. $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}+\tan^{-1}\frac{1}{21}+\cdots$

[Hint. $T_1=\tan^{-1}\dfrac{1}{1+1.2}+\tan^{-1}2-\tan^{-1}$.]

22. $\tan^{-1}\dfrac{1}{2.1^2}+\tan^{-1}\dfrac{1}{2.2^2}+\tan^{-1}\dfrac{1}{2.3^2}+\cdots$

[Hint. $T_n=\tan^{-1}\dfrac{1}{2n^2}=\tan^{-1}\dfrac{2}{1+(4n^2-1)}$
$=\tan^{-1}\dfrac{(2n+1)-(2n-1)}{1+(2n-1)(2n+1)}=\tan^{-1}(2n+1)-\tan^{-1}(2n-1)$].

13·4. Sum to n terms of the series.

$u_0\cos\alpha+u_1\cos(\alpha+\beta)+u_2\cos(\alpha+2\beta)+\cdots$(1),
$u_0\sin\alpha+u_1\sin(\alpha+\beta)+u_2\sin(\alpha+2\beta)+\cdots$(2),
where u_0, u_1, u_2 ..form an arithmetical progression.

Let $S=u_0\cos\alpha+u_1\cos(\alpha+\beta)+u_2\cos(\alpha+2\beta)+\cdots$
$+u_{n-1}\cos(\alpha+\overline{n-1}\beta)$.

$2\cos\beta.\ S=u_0[\cos(\alpha+\beta)+\cos(\alpha-\beta)]+u_1[\cos(\alpha+2\beta)+\cos\alpha]$
$+u_2[\cos(\alpha+3\beta)+\cos(\alpha+\beta)]+\cdots+u_{n-1}[\cos(\alpha+n\beta)$
$+\cos(\alpha+\overline{n-2}\beta)]$.

$\therefore\ 2(1-\cos\beta)\,S=(2u_0-u_1)\cos\alpha+(2u_1-u_0-u_2)\cos(\alpha+\beta)$
$+(2u_2-u_1-u_3)\cos(\alpha+2\beta)+\cdots$
$+(2u_{n-2}-u_{n-3}-u_{n-1})\cos(\alpha+\overline{n-2}\,\beta)$
$-u_0\cos(\alpha-\beta)-u_{n-1}\cos(\alpha+n\beta)$.
$+(2u_{n-1}-u_{n-2})\cos(\alpha+\overline{n-1}\beta)$.

Since $u_0, u_1, u_2\cdots$form an A.P.,
$2u_r=u_{r-1}+u_{r+2},\quad r=0,\ 1,\ldots n-1,$
$\therefore\ 2(1-\cos\beta)S=(2u_0-u_1)\cos\alpha-u_0\cos(\alpha-\beta)$
$+(2u_{n-1}-u_{n-2})\cos(\alpha+\overline{n-1}\beta)-u_{n-1}\cos(\alpha+n\beta)$.
S is found by dividing both sides of this result by
$2(1-\cos\beta)S$.

The sum of the second series may be obtained by a similar method, or by differentiating the first series with respect to a.

Examples XIII (c)

Sum to n terms the following series

1. $1+2\sin x+3\sin 2x+4\sin 3x+\ldots+n\sin \overline{n-1}\,x$.
2. $\cos x+2\cos 2x+3\cos 3x+\ldots+n\cos nx$.
3. $\sin x+2\sin 2x+3\sin 3x+\ldots+n\sin nx$.
4. $3\sin x+5\sin 2x+7\sin 3x+\ldots+(2n+1)\sin nx$.

13·5. Sum to n terms of the series.

$$C_n=\cos a+x\cos(a+\beta)+x^2\cos(a+2\beta)+\ldots$$
$$S_n=\sin a+x\sin(a+\beta)+x^2\sin(a+2\beta)+\ldots$$

If both sides of the equation C_n be multiplied by $1-2x\cos\beta+x^2$ and the terms containing like powers of x be collected, all the intermediate terms will be found to vanish. For instance, the coefficient of x^2 will be

$$\cos(a+2\beta)-2\cos(a+\beta)\cos\beta+\cos a,$$

which is zero. Similarly the coefficient of x^r in the product will be

$$\cos(a+r\beta)-2\cos\beta\cos(a+\overline{r-1}\beta)+\cos(a+\overline{r-2}\beta)$$

and this is also zero. Hence

$$(1-2x\cos\beta+x^2)\,C_n=\cos a-x\cos(a-\beta)-x^n\cos(a+n\beta)+x^{n+1}\cos(a+\overline{n-1}\beta),$$

$$\therefore\ C_n=\frac{\cos a-x\cos(a-\beta)}{1-2x\cos\beta+x^2}-x^n\frac{\cos(a+n\beta)-x\cos(a+\overline{n-1}\beta)}{1-2x\cos\beta+x^2}.$$

It will similarly be found that

$$S_n=\frac{\sin a-x\sin(a-\beta)}{1-2x\cos\beta+x^2}-x^n\frac{\sin(a+n\beta)-x\sin(a+\overline{n-1}\beta)}{1-2x\cos\beta+x^2}.$$

The factor $1-2x\cos\beta+x^2$ is called the *scale of the recurring series.*

Each result may be obtained from the other by differentiation with respect to a.

13·51. Sum to infinity of the series in 13·5.

If $|x|<1$, $x^n\to 0$, as $n\to\infty$. Also

$|\cos(a+n\beta)|\leqslant 1$, $|\sin(a+n\beta)|\leqslant 1$, for every n.

$$\therefore\ \operatorname*{Lt}_{n\to\infty} C_n=\frac{\cos a-x\cos(a-\beta)}{1-x\cos\beta+x^2},$$

and

$$\operatorname*{Lt}_{n\to\infty} S_n=\frac{\sin a-x\sin(a-\beta)}{1-x\cos\beta+x^2}.$$

13·6. Use of the exponential values of sine and cosine of an angle in the summation of series.

Let us consider the series of art. 13·5.

Put $\cos\beta+i\sin\beta$, or $e^{i\beta}=z$;

then $$C_n+iS_n=e^{i\alpha}[1+xz+x^2z^2+\ldots\ldots]$$
$$=e^{i\alpha}\frac{1-x^nz^n}{1-xz}$$
$$=e^{i\alpha}\frac{(1-x^nz^n)\left(1-\frac{x}{z}\right)}{(1-xz)\left(1-\frac{x}{z}\right)}$$
$$=e^{i\alpha}\frac{1-x^nz^n-\frac{x}{z}+x^{n+1}z^{n-1}}{1-x\left(z+\frac{1}{z}\right)+x^2}.$$

If we now replace z by $\cos\beta+i\sin\beta$, and $\frac{1}{z}$ by $\cos\beta-i\sin\beta$:

$$C_n+iS_n$$
$$=(\cos\alpha+i\sin\alpha)\{1-x^n(\cos n\beta+i\sin n\beta)-x(\cos\beta-i\sin\beta)$$
$$+x^{n+1}(\cos\overline{n-1}\beta+i\sin\overline{n-1}\beta)\}/(1-2x\cos\beta+x^2)$$
$$=\{\text{cis}\,\alpha-x^n\,\text{cis}\,(\alpha+n\beta)-x\,\text{cis}\,(\alpha-\beta)$$
$$+x^{n+1}\text{cis}\,(\alpha+\overline{n-1}\beta)\}/(1-2x\cos\beta+x^2).$$

$$\therefore\quad C_n=[\cos\alpha-x\cos(\alpha-\beta)-x^n\cos(\alpha+n\beta)$$
$$+x^{n+1}\cos(\alpha+\overline{n-1}\beta)]/(1-2x\cos\beta+x^2),$$

and $$S_n=\{\sin\alpha-x\sin(\alpha-\beta)-x^n\sin(\alpha+n\beta)$$
$$+x^{n+1}\sin(\alpha+\overline{n-1}\beta)\}/(1-2x\cos\beta+x^2).$$

13·61. The following is a simpler illustration.

We take the sine and cosine series of Art. 13·11.

Let $C=\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots+\cos(\alpha+\overline{n-1}\beta)$,

and $S=\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\ldots+\sin(\alpha+\overline{n-1}\beta)$;

put $z=\cos\beta+i\sin\beta=e^{i\beta}$.

Then $C+iS=e^{i\alpha}[1+z+z^2+\ldots n\text{ terms}]$
$$=e^{i\alpha}\left(\frac{1-z^n}{1-z}\right)$$
$$=e^{i\alpha}\frac{(1-z^n)\left(1-\frac{1}{z}\right)}{1-\left(z+\frac{1}{z}\right)+1},$$

[multiplying num. and den. by $1-\frac{1}{z}$.

$$=e^{i\alpha}\frac{1-z^n-\frac{1}{z}+z^{n-1}}{2-\left(z+\frac{1}{z}\right)},$$

Substituting $z=\cos\beta+i\sin\beta$, $\frac{1}{z}=\cos\beta-i\sin\beta$,

$$C+iS=e^{i\alpha}\frac{1-\text{cis}\,n\beta-\text{cis}\,(-\beta)+\text{cis}\,\overline{n-1\beta}}{2\,(1-\cos\beta)}$$

$$=\frac{\text{cis}\,\alpha-\text{cis}\,(\alpha+n\beta)-\text{cis}(\alpha-\beta)+\text{cis}(\alpha+\overline{n-1}\beta)}{2(1-\cos\beta)}$$

$$=e^{i(\alpha+\overline{n-1}\beta/2)}\frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$$

$$=\frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\{\cos\,(\alpha+\overline{n-1}\,\frac{\beta}{2})+i\sin\,(\alpha+\overline{n-1}\,\frac{\beta}{2})\}.$$

Equating the real and imaginary parts, we get the values of C and S.

13·62. As another illustration we take the series of art. 13·5.

$$C_n=u_0\cos\alpha+u_1\cos(\alpha+\beta)+\cdots+u_{n-1}\cos(\alpha+\overline{n-1}\beta),$$
$$S_n=u_0\sin\alpha+u_1\sin(\alpha+\beta)+\cdots+u_{n-1}\sin(\alpha+\overline{n-1}\beta).$$
$$C_n+iS_n=e^{i\alpha}[u_0+u_1z+u_2z^2+\cdots+u_{n-1}z^{n-1}],$$

where $z=\cos\beta+i\sin\beta$.

$$\therefore\quad z(C_n+iS_n)=e^{i\alpha}[u_0z+u_1z^2+u_2z^3+\cdots+u_{n-2}z^{n-1}+u_{n-1}z^n].$$

Now we subtract second from the first.

Since $u_0, u_1, \ldots$ form an A.P, $u_1-u_0=u_2-u_1=\cdots=u_n-u_{n-1}$.

$$\therefore\ (C_n+iS_n)(1-z)=e^{i\alpha}[u_0-u_{n-1}z^n+(u_1-u_0)(z+z^2+\ldots+z^{n-1})$$
$$=e^{i\alpha}\left[u_0-u_{n-1}\ z^n+(u_1-u_0)\frac{z\,(1-z^{n-1})}{1-z}\right].$$

Multiplying both sides by $1-\frac{1}{z}$,

$$(C_n+iS_n)(1-z)\left(1-\frac{1}{z}\right)=e^{i\alpha}\left[u_0\left(1-\frac{1}{z}\right)-u_{n-1}(z^n-z^{n-1})\right.$$
$$\left.-(u_1-u_0)(1-z^{n-1})\right]$$

$$\therefore \left\{2-\left(z+\frac{1}{z}\right)\right\}(C_n+iS_n)=e^{i\alpha}[(2u_0-u_1)-u_0\frac{1}{z}-u_{n-1}(z^n-z^{n-1})$$
$$+(u_{n-1}-u_{n-2})z^{n-1}].$$

Since $u_1-u_0=u_{n-1}-u_{n-2}$.

Replacing z by $\cos\beta+i\sin\beta$,

$$\therefore \quad 2(1-\cos\beta)(C_n+iS_n)=[(2u_0-u_1)\operatorname{cis}\alpha-u_0\operatorname{cis}(\alpha-\beta)$$
$$+(2u_{n-1}-u_{n-2})\operatorname{cis}(\alpha+n-1\beta)-u_{n-1}\operatorname{cis}(\alpha+n\beta)].$$

$$\therefore \quad 2(1-\cos\beta)C_n=(2u_0-u_1)\cos\alpha-u_0\cos(\alpha-\beta)$$
$$+(2u_{n-1}-u_{n-2})\cos(\alpha+n-1\beta)-u_{n-1}\cos(\alpha+n\beta),$$

and $2(1-\cos\beta)S_n=(2u_0-u_1)\sin\alpha-u_9\sin(\alpha-\beta)$
$$+(2u_{n-1}-u_{n-2})\sin(\alpha+n-1\beta)-u_{n-1}\sin(\alpha+n\beta).$$

13·7. The methods used above in summing up series of different kinds may be classified as follows :—

1. *Method of Differences.*—Every term a_n may be put as the difference of two functions in the form $\phi(n)-\phi(n-1)$. The series will thus become a telescopic series, the intermediate terms cancelling in pairs. This may be done sometimes by multiplying each term by a constant factor.

2. *Method of Recurring Series.* Multiplication of the series by a certain factor, called the scale of the series, removes the intermediate terms, as in 13·5.

3. *Method of Exponenital Values* :—By expressing each term in the expeonential form, the series may be reduced to one of the algebraic series—Geometrical Progression, Arithmetico-geometric series, Exponential series, Binomial series, or Logarithmic series etc. This method is in essence the same as that of summing up a pair of sine and cosine trigonometrical series by the use of complex numbers. The method (3) will be further illustrated by examples given in article 13·6.

13·8. Infinite Series. Infinite series can also be summed up by the methods outlined above. Care is to be taken that convergence is established in each case. It is outside the scope of this book to deal with any rigour with infinite Binomial, Logarithmic, and Exponental series etc., involving complex numbers. The expansions and their convergence will, therefore, generally be assumed, and the limitations under which they hold, stated.

(*i*) **Series of Complex terms :—**

If $z_n=x_n+iy_n$, Σz_n coges. if Σx_n and Σy_n converge separately (as defined already in 12·6),

Putting $x_n+iy_n=r_n(\cos\theta_n+i\sin\theta_n)$, it follows that $\Sigma r_n\cos\theta_n$ and $\Sigma r_n\sin\theta_n$ converge, if Σr_n, *i.e.*, $\Sigma|z_n|$ coges.

(*ii*) **The Geometrical Series.** The sum of the finite series

$$1+z+z^2+\cdots+z^n \text{ is } \frac{1-z^n}{1-z}=\frac{1}{1-z}-\frac{z^n}{1-z}.$$

Now let $z=x+iy=r(\cos\theta+i\sin\theta)$,

then $z^n=r^n(\cos\theta+i\sin\theta)^n=r^n(\cos n\theta+i\sin n\theta)$.

$$\therefore \operatorname*{Lt}_{n\to\infty}\frac{r^n(\cos n\theta+i\sin n\theta)}{1-r\cos\theta-i\,r\sin\theta}=0, \text{ where } r, \text{ or } |z|<1.$$

(*iii*) **The Binomial Series**

$$1+nz+\frac{n(n-1)}{2!}z^2+\frac{n(n-1)(n-2)}{3!}z^3+\cdots \qquad (1)$$

Case 1. When n is a positive integer, there is no difficulty, for the ordinary proof holds.

Case 2. When n is negative or fractional, and $|z|<1$, the series (1) coges. and its sum is one of the values of $(1+z)^n$, which is called the *principal value*.

Case 3. When n is negative, or fractional, and $|z|=1$, the theorem is true if (1) n is $+ve$, or (2) n is a negative proper fraction, and z is not equal to -1.

(*iv*) **The Exponential Series**

$$\mathrm{E}(z)=1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots$$

The series is seen to be absolutely convergent for all values of z, real or complex.

(*v*) **The Logarithmic Series :—**

$$z-\frac{z^2}{2}+\frac{z^3}{3}-\cdots$$

When z is real, this series is known to be the expansion of $\log(1+z)$, and it coges. if z lies between $+1$ and -1.

This is also true when $z=+1$, but not when $z=-1$.

When z is complex, and $|z|<1$, the series coges. and its sum is $\log(1+z)$, the principal value of $\log(1+z)$.

If $|z|=1$, the series holds except when am $z=(2n+1)\pi$.

Example 1. Sum up the infinite series :—

$S=\cos 2\theta+\frac{1}{3}\cos 6\theta+\frac{1}{5}\cos 10\theta+\cdots$

Putting the exponential values for cosines,

$$2S=(e^{2i\theta}+e^{-2i\theta})+\tfrac{1}{3}(e^{6i\theta}+e^{-6i\theta})+\tfrac{1}{5}(e^{10i\theta}+e^{-10i\theta})+\cdots$$

$$=(e^{2i\theta}+\tfrac{1}{3}e^{6i\theta}+\tfrac{1}{5}e^{10i\theta}+\cdots)+(e^{-2i\theta}+\tfrac{1}{3}e^{-6i\theta}+\tfrac{1}{5}e^{-10i\theta}+\cdots)$$

$$\cdots(1)$$

Now $\log\frac{1+z}{1-z}=2\left(z+\frac{z^3}{3}+\frac{z^5}{5}+\cdots\right)$,

when $|z|=1$, and am $z=2n\pi$.

Hence from (1)

$$2S=\tfrac{1}{2}\log\frac{1+e^{2i\theta}}{1-e^{2i\theta}}+\tfrac{1}{2}\log\frac{1+e^{-2i\theta}}{1-e^{-2i\theta}}$$

$$=\tfrac{1}{2}\log\frac{(1+e^{2i\theta})(1+e^{-2i\theta})}{(1-e^{2i\theta})(1-e^{-2i\theta})}=\tfrac{1}{2}\log\frac{2^2\cos^2\theta}{2^2\sin^2\theta}.$$

$$\therefore\quad S=\tfrac{1}{2}\log\cot\theta.$$

Note. If the sum of the corresponding sine-series also were required, the method of "C+iS" would have been useful.

Example 2. Sum the two series :—

$$C=1+\tfrac{1}{2}\cos\alpha+\frac{1.3}{2.4}\cos 2\alpha+\frac{1.3.5}{2.4.6}\cos 3\alpha+\cdots\text{ad inf.}$$

and $$S=\tfrac{1}{2}\sin\alpha+\frac{1.3}{2.4}\sin 2\alpha+\frac{1.3.5}{2.4.6}\sin 3\alpha+\ldots\text{ad inf.}$$

Now $C+iS=1+\frac{1}{2}e^{i\alpha}+\frac{\frac{1}{2}\cdot\frac{3}{2}}{2\,!}e^{2i\alpha}+\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3\,!}e^{3i\alpha}+\ldots$ad inf.

$$=(1-e^{i\alpha})^{-\frac{1}{2}},\text{ when }\left|-e^{i\alpha}\right|<1.$$

[It may be noted that $|\cos+i\sin\alpha|<1$ if $\alpha\neq k\pi$, and if $\alpha=(2k+1)\pi$, $|z|=+1$, and the formula is true.]

Hence $C+iS=\{\overline{1-\cos\alpha}-i\sin\alpha\}^{-\frac{1}{2}}$

$$=\left\{2\sin\frac{\alpha}{2}\left(\sin\frac{\alpha}{2}-i\cos\frac{\alpha}{2}\right)\right\}^{-\frac{1}{2}}$$

$$=\frac{1}{\sqrt{2\sin\frac{\alpha}{2}}}\left\{\cos\frac{\alpha-\pi}{2}+i\sin\frac{\alpha-\pi}{2}\right\}^{-\frac{1}{2}}$$

$$=\frac{1}{\sqrt{2\sin\frac{\alpha}{2}}}\left\{\cos\frac{\pi-\alpha}{4}+i\sin\frac{\pi-\alpha}{4}\right\}.$$

Now equate the real and the imaginary parts.

Examples XIII (*d*)

Sum the following series, each to n terms :—

1. $\cos\theta\cos(\theta+\alpha)+\cos(\theta+\alpha)\cos(\theta+2\alpha)$
$+\cos(\theta+2\alpha)\cos(\theta+3\alpha)+\cdots$

2. $\sin(p+1)\theta\cos\theta+\sin(p+2)\theta\cos 2\theta$
$+\sin(p+3)\theta\cos 3\theta+\cdots$

3. $1+2\cos\theta+3\cos 2\theta+\cdots$

4. $3.4\sin\alpha+4.5\sin 2\alpha+\cdots$

5. $1.2\cos\theta+2.3\cos 2\theta+3.4\cos 3\theta+\cdots$

6. Show that

$$\frac{\sin\theta-\sin 2\theta+\sin 3\theta-\cdots\text{to } n \text{ terms}}{\cos\theta-\cos 2\theta+\cos 3\theta-\cdots\text{to } n \text{ terms}}=\tan\frac{n+1}{2}(\pi+\theta).$$

7. $$\frac{\sin A}{\cos A\cos 2A}+\frac{\sin A}{\cos 2A\cos 3A}+\frac{\sin A}{\cos 3A\cos 4A}+\cdots\text{to } n \text{ terms}$$
$$=\tan(n+1)A-\tan A$$

8. Show that

$$\sinh x+n\sinh 2x+\frac{n(n-1)}{2!}\sinh 3x+\cdots\text{to } n+1 \text{ terms}$$
$$=2^n\cosh^n\frac{x}{2}\sinh\left(\frac{n}{2}+1\right)x.$$

9. With the aid of complex numbers, sum up the series :—

(*i*) $\cos\theta+x\cos 2\theta+x^2\cos 3\theta+\cdots$ to n terms

(*ii*) $\sin\theta+x\sin 2\theta+x^3\sin 3\theta+\cdots$ to n terms

10. Sum to n terms the series :—

(*i*) $C=\cos\theta\cos\theta+\cos^2\theta\cos 2\theta+\cos^3\theta\cos 3\theta+\cdots$

(*ii*) $S=\cos\theta\sin\theta+\cos^2\theta\sin 2\theta+\cos^2\theta\sin 3\theta+\cdots$

[Hint. $C+iS=z\cos\theta+z^2\cos^2\theta+\cdots$ where $z=\cos\theta+i\sin\theta$.]

11. Sum to n terms :—

(*i*) $\frac{\sin\theta}{\cos\theta}+\frac{\sin 2\theta}{\cos^2\theta}+\frac{\sin 3\theta}{\cos^3\theta}+\cdots$

(*ii*) $\frac{\cos\theta}{\cos\theta}+\frac{\cos 2\theta}{\cos^2\theta}+\frac{\cos 3\theta}{\cos^3\theta}+\cdots$

12. Sum to $n+1$ terms the series :—

$$1+n\cos\theta+\frac{n(n-1)}{2!}\cos 2\theta+\frac{n(n-1)(n-2)}{3!}\cos 3\theta+\cdots$$

13. Sum the series :—

$$\tan\frac{x}{2}\sec x+\tan\frac{x}{2^2}\sec\frac{x}{2}+\cdots+\tan\frac{x}{2^n}\sec\frac{x}{2^{n-1}}.$$

14. Sum the series

$$\sin\theta\sin^2\frac{\theta}{2}+2\sin\frac{\theta}{2}\sin^2\frac{\theta}{4}+4\sin\frac{\theta}{4}\sin^2\frac{\theta}{8}+\cdots\text{to } n \text{ terms.}$$

Sum to infinity the following series :—

15. $\cos\theta+\frac{\cos\theta}{1}\cos 2\theta+\frac{\cos^2\theta}{2!}\cos 3\theta+\frac{\cos^3\theta}{3!}\cos 4\theta+\cdots$

16. $\cos\theta+\frac{\sin\theta}{1}\cos 2\theta+\frac{\sin^2\theta}{2!}\cos 3\theta+\cdots$

17. $2\cos\theta+\frac{3}{2}\cos^2\theta+\frac{4}{3}\cos^3\theta+\frac{5}{4}\cos^4\theta+\cdots$

18. $x \sin\theta - \frac{x^2 \sin 2\theta}{2} + \frac{x^3 \sin 3\theta}{3} - \cdots$

19. $\cos\theta - \frac{1}{2}\cos 2\theta + \frac{1}{3}\cos 3\theta + \cdots$

20. $\sin\theta - \frac{\sin 2\theta}{2!} + \frac{\sin 3\theta}{3!} - \cdots$ **21.** $1 - \frac{\cos 2\theta}{2!} + \frac{\cos 4\theta}{4!} - \cdots$

22. $\sin\theta + \frac{1}{2}\sin 2\theta + \frac{1}{2^2}\sin 3\theta + \cdots$

23. $\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{2^2}\cos 3\theta + \cdots$

24. $1 + x\cosh\alpha + x^2\cosh 2\alpha + x^3\cosh 3\alpha + \cdots$

25. $x\sinh\alpha + x^3\sinh 2\alpha + x^3\sinh 3\alpha + \cdots$

26. $x\sin\theta + \frac{x^2\sin 2\theta}{2!} + \frac{x^3\sin 3\theta}{3!} + \cdots$

27. $1 - x\cos\theta + \frac{x^2}{2!}\cos 2\theta - \cdots$

28. $\sin\alpha + x\sin(\alpha+\beta) + \frac{x^2}{2!}\sin(\alpha+2\beta) + \cdots$

29. $\cos\alpha + x\cos(\alpha+\beta) + \frac{x^2}{2!}\cos(\alpha+2\beta) + \cdots$

30. $\sin\alpha + \frac{1}{2}\sin 3\alpha + \frac{1.3}{2.4}\sin 5\alpha + \cdots$

31. $n\cos\alpha + \frac{n(n+1)}{1.2}\cos 2\alpha + \frac{n(n+1)(n+2)}{1.2.3}\cos 3\alpha + \cdots$

32. $\frac{1}{2}\sin\alpha - \frac{1}{2.4}\sin 2\alpha + \frac{1.3}{2.4.6}\sin 3\alpha - \cdots$

33. $1 - \cos\alpha\cos\beta + \frac{\cos^2\alpha}{2!}\cos 2\beta - \frac{\cos^3\alpha}{3!}\cos 3\beta + \cdots$

34. $\sinh u + \frac{\sinh 2u}{2!} + \frac{\sinh 3u}{3!} + \cdots$

35. $x\sin\alpha + \frac{1}{4}x^3\sin^3\alpha + \frac{1.3}{4.8}x^5\sin 5\alpha + \frac{1.3.5}{4.8.12}x^7\sin 7\alpha + \cdots$

36. (*i*) $\frac{x^2\sin 2\theta}{2!} + \frac{x^4\sin 4\theta}{4!} + \cdots$

(*ii*) $1 + \frac{x^2\cos 2\theta}{2!} + \frac{x^4\cos 4\theta}{4!} + \cdots$

37. $\sin\theta - \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5 - \cdots$ where $-\frac{\pi}{2} < 0 < \frac{\pi}{2}$.

38. Sum the series $c\cos\alpha + \frac{1}{3}c^3\cos 3\alpha + \frac{1}{5}c^5\cos 5\alpha + \cdots$ ad inf. *[P.U. 1945]*

ANSWERS

Examples I (*a*) (Pp. 4-5)

1. $a=11$, $b=-6$. 2. $a=1$, $b=0$, $c=2$. 4. n is an odd integer, or a multiple of 4.

Examples I (*b*) (Pp. 6-7)

1. (*i*) $x^4+2x^3-25x^2-26x+120=0$.
   (*ii*) $25x^4+120x^3-66x^2-184x+105=0$.
2. $x^2+(m+n)x+m^2-mn+n^2=0$.
3. $x^3-3mnx-(m^3+n^3)=0$. 5. -61. 6. $8, 8, -\frac{1}{2}$.
7. $-4, 1, 2, 3$. 8. $1, 2, 3, -4$. 9. $5, -19, 21, -45$.
10. $Q=4x^4-2x^3+2x^2-3x+3$, $R=-5$.
11. $2x^3-9x^2+18x-29$, $R=53$.

Examples I (*c*) (Pp. 10-12)

1. $x^3-8x^2+26x-28$. 2. $x^4-2x^3-11x^2+6x+2=0$.
3. (*i*) $3, -1, 3\pm\sqrt{30}$; (*ii*) $-1, -6, 3\pm i\sqrt{2}$.
4. $1\pm3i, -2$. 5. $(x+2)(x^2+2x+5)$. 6. ±1.
7. $-2, 3$. 8. $\pm\sqrt{2}\pm\sqrt{3}, 2\pm\sqrt{5}$. 9. $\pm\sqrt{2}\pm\sqrt{5}, \frac{4}{3}$.
10. $\pm\sqrt{2}+\sqrt{-1}, 3(1\pm\sqrt{-7})/4$.
11. $4, -1+\sqrt[3]{3}, -1+\omega\sqrt[3]{3}, -1+\omega^2\sqrt[3]{3}$.
13. $\sqrt{\frac{3}{2}}+\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}-\sqrt{\frac{3}{2}}$. 17. n.
18. $n(n+1)x^2-2(n+1)x+2=0$. 19. $-4, 2$.
20. $0, 1, \frac{-3}{2}, \frac{-5}{2}$; $0, 1, \frac{-3}{2}, \frac{-5}{3}$.

Examples I (*d*) (Pp. 13-19)

2. (*i*) $\pm\frac{1}{5}, -2$; (*ii*) $\pm\frac{1}{2}, 2, 4$; (*iii*) $\pm\sqrt{-\frac{1}{3}}, \frac{1\pm\sqrt{-15}}{4}$.
4. $k=54$, roots $3, 3, 6$; $k=50$, roots $5, 5, 2$.
5. $1, 1, \frac{5}{2}$; $2, 2, \frac{1}{2}$. 6. $r^2+4q^3=0$.
7. $1, 1, -2$; $-1, -1, 2$. 8. $\frac{3\pm\sqrt{5}}{2}, \frac{5\pm\sqrt{5}}{2}$.
9. $p, q=1\pm\sqrt{17}$; roots are $\frac{7\pm\sqrt{17}}{2}, \frac{3\pm\sqrt{17}}{2}$
10. $-5, 4, -2, 1$. 12. $2\pm i, 3\pm2i$.
14. $-1\pm i, \frac{1\pm\sqrt{-7}}{2}$. 15. $\pm\sqrt{3}, 1\pm i\sqrt{6}$.

16. $\pm\frac{i}{2}$; $-1\pm\sqrt{-2}$. 17. $\frac{3\pm\sqrt{-6}}{5}$, $\pm\frac{5}{3}$.

18. $\pm\frac{i}{\sqrt{3}}$, $\frac{1\pm\sqrt{-15}}{4}$. 19. (*ii*) $b^2=ac$, $a^2d=b^3$, $a^3c=b^4$.

21. 3, 3, 4, 4. 22. $-\frac{3}{2}$, -1, $-\frac{1}{2}$. 23. 1, 4, 7.

24. $-\frac{1}{2}$, $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$. 25. $\frac{1}{4}$, $\frac{1}{2}$, 1. 26. $\frac{-2}{9}$; $\frac{2}{3}$, -2.

27. $\frac{3}{2}$, 3, 6, 12. 28. 1, -3, 9, -27.

29. $\pm 2i$. 30. 6, 4, -1. 36. $\frac{3}{2}$, $\frac{3}{2}$, $\frac{3}{2}$, $\frac{-9\pm3\sqrt{5}}{4}$.

39. $a_1{}^3-4a_0a_1a_2+8a_0{}^2a_3=0$; $-1\ \frac{-1\pm\sqrt{-7}}{2}$.

40. 2, 6, 1, 3. 43. $x=abc$, $y=-\Sigma bc$, $z=\Sigma a$.

Examples II (*a*) (P. 23)

3. (*i*) $\cos(x+y)$ (*ii*) $2y^3$.
4. (*i*) -315. (*ii*) -168. (*iii*) 360.

Examples II (*b*) (Pp. 30-31)

1. (*i*) 6; (*ii*) 14; (*iii*) 15; (*iv*) 19. 2. $+ve$.
4. (*i*) -8; (*ii*) 0. 6. 0. 7. -27.
9. $a^2l^2+b^2m^2+c^2n^2-2bcmn-2canl-2ablm$.
10. $a^4+(l^2+m^2+n^2+\alpha^2+\beta^2+\gamma^2)a^2+(l\gamma+m\beta+n\alpha)^2$.
11. $p^4+q^4+r^4+s^4-2p^2q^2-2q^2r^2-2r^2s^2-2s^2p^2-2p^2r^2-2q^2s^2-8pqrs$.

Examples II (*c*) (Pp. 36-47)

6. 6. 15. -1, -1, -2.
19. $-(Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm)+d\Delta$, where A, B, ...are co-factors of the corresponding small letters in the determinant Δ of Ex. 18.
24. -8^5. 25. 0. 33. $abc\,(b-c)\,(c-a)\,(a-b)$.
40. $(a+b+c)^3$.

Examples II (*d*) (Pp. 50-51)

1. (*a*) $x=4$, $y=2$, $z=3$; (*b*) $x=1$, $y=2$, $z=3$.
(*c*) $x=\frac{1}{3}$, $y=1$, $z=-\frac{1}{3}$.
2. $x=\frac{1}{5}$, $y=\frac{3}{5}$, $z=-\frac{1}{5}$, $w=-\frac{2}{5}$.
3. $x:y:z:1=a\,(b-c)^3:b\,(c-a)^3:c\,(a-b)^3:(b-c)(c-a)\,(a-b)$.
4. $x=\frac{(k-b)(k-c)}{(a-b)(a-c)}$, etc.

5. $x=\dfrac{\lambda(\lambda-\beta)(\lambda-\gamma)}{(\alpha-\beta)(\alpha-\gamma)}$, and two similar values for y and z.

6. Consistent. 8. $x:y:z=1:3:-10$.

10. $\begin{Vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \end{Vmatrix}$ 11: $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \div (ab-h^2)$.

12. $k=-(l_1m_2n_3p_4)\div(l_2m_3n_4)$. 13. $1+yz+zx+xy=0$.

Examples II (*e*) (Pp. 54-56)

6. $\begin{vmatrix} a^2+h^2+g^2 & ah+bh+fg & ag+fh+cg \\ ah+bh+fg & h^2+b^2+f^2 & gh+bf+cf \\ ag+fh+cg & gh+bf+cf & g^2+f^2+c^2 \end{vmatrix}$. 8. $\Delta=\pm 1$.

Examples II (*f*) (Pp. 62-63)

2. $(x-a)^4(x+4a)$.

3. $-(a+b+c-d)(a+b+d-c)(a-b+c+d)(-a+b+c+d)$.

4. $(a+b+c+d)(a-b+c-d)(a-c+i\sqrt{b-d})(a-c-i\sqrt{b-d})$.

5. $-(x+y+z)(y+z-x)(z+x-y)(x+y-z)$.

Examples II (*g*) (Pp. 63-71)

4. $x=-\frac{1}{9}\frac{1}{7}$. 6. $x=0, 10$.

21. $x=\dfrac{a(a+b+c)}{(a-b)(a-c)}$, etc. If $ab+bc+ca=0$, then $x=y=z=1$.

22. $\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix} \div \begin{vmatrix} u & w' & v' & a \\ w' & v & u' & b \\ v' & u' & w & c \\ a & b & c & 0 \end{vmatrix}$.

24. $t=0, 3$; $x=y=z$; when $t=3$, the equations are identical.

25. $a^2c^2f^2h^2$. 26. $x=a$, $y=b$, $z=c$.

Examples III (*a*) (Pp. 72-75)

1. $x^3+\dfrac{6b}{a}x^2+\dfrac{3(3b^2+ac)}{a^2}x-\dfrac{ad-9bc}{a^2}=0$.

2. $(c-ab)y^3+(3c-2ab+a^3)y^2+(3c-ab)y+c=0$.

3. $ry^3-qy^2-1=0$, $(y-q)^3+q(y-q)^2+pr(y-q)+r^2=0$.

4. $(27b^3-36abc+8a^2d)y^3+(27b^3-36abc+12a^2d)y^2$
$+3a(2ad-3bc)y+a^2d=0.$

5. $f\left\{\frac{-4by}{a(2y+1)}\right\}=0$, if $f(x)\equiv ax^4+4bx^3+\ldots\ldots=0.$

6. $y^3-28y^2+245y-650=0.$

7. $c^2y^3-ac(c+1)y^2+(c+1)^2by-(c+1)^3=0.$

11. $1, 2, 1\pm\sqrt{-2}.$

12. $1, -\frac{2}{3}, 3.$ 13. $\frac{-3\pm\sqrt{-3}}{4}, \frac{-3}{2}; \frac{-3\pm\sqrt{-3}}{4}, \frac{-15}{2}.$

14. $\frac{-2}{3}, 2\pm i; -5, 3\pm 2i.$ 15. $\frac{3}{2}, \frac{-3}{2}, 2, 2, -4.$

16. $2, 5, \pm i.$

Examples III (*b*) (Pp. 76-77)

1. (*i*) $7x^6+8x^5+9x^4+x^3-x^2+3x+1=0.$
(*ii*) $x^7+8x^4+3x^2-7x-9=0.$

3. $x^4-7x^3+48x^2-252x+432=0.$

4. $x^3-4x^2-18x-648=0.$

5. $x^4-3x^3+4x^2-27x+108=0.$

6. $5x^4+2x^3+x^2-2x-1=0.$

7. $x^5-50x^3-200x^2-56x+400=0.$

8. $x^4+2x^3-6x-6=0$; $1:-4.$

9. $m=4$, $x^3-6x^2-x+2=0.$

10. $m=30$, $x^4-25x^3+375x^2-11{,}700=0.$

11. $x^3-3x^2+3x+12=0.$

Examples III (*c*) (P. 78)

1. $\frac{2}{3}, \frac{2}{5}, \frac{2}{7}.$ 2. $-\frac{1}{5}, -\frac{1}{2}, 1, \frac{1}{4}.$

Examples III (*d*) (Pp. 80-82)

2. $x^4+20x^3+153x^2+526x+689=0.$

3. $3x^5-60x^4+475x^3-1860x^2+3600x-2745=0.$

4. $4x^5-40x^4+158x^3-308x^2+303x-129=0.$

5. $3x^3-76x-156=0.$ 6. (*i*) $c=1$; (*ii*) $c=-1, -3.$

7. $2x^4-46x^2-121x-85=0.$

9. $2, 3, 55.$ 10. $-\frac{1}{2}, -1, -3, -\frac{7}{3}.$

11. The transformed equation is $y^4-24y^2-52=0$; $-4\pm\sqrt{26}$, $-4\pm\sqrt{-2}.$ 12. $-7, -3, -5\pm\sqrt{3}.$

Examples III (*e*) (Pp. 84-85)

3. $-4, \frac{1\pm 3\sqrt{-3}}{2}.$ 4. $h=2, \frac{9}{2}$; $x^4-5x^3+2x-2=0.$

$x^4+5x^3-\frac{113}{4}x-\frac{537}{16}=0.$ 6 $-6, -2, -7, -1.$

7. $2\pm\sqrt{2}$, $2\pm\sqrt{3}$. 8. $z^3+9z+6=0$.

9. $x^4+\frac{9}{2}x^2+x-\frac{3}{16}=0$. 10. $a_0^2a_3-3a_0a_1a_2+2a_1^3=0$.

Examples III (f) (Pp. 80-88)

2. $-2,849$. 3. 795. 5. 518. 6. -54. 7. $100a^2$.

Examples III (g) (Pp. 90-95)

1. $\frac{8}{7}$. 3. $4(4a_1^2-3a_0a_2)$; $4(4a_3^2-3a_2a_4)$. 6. $c-ab$.

7. $27H$, $-27G$. 8. $96H$, $-256G$, $256(a_0^2I-3H^2)$.

9. $\frac{4}{a_0^3}[24a_1^2a_2-18a_0a_2^2-4a_0a_1a_3+a_0^2a_4]$. 10. $\frac{12}{a_0^2}(a_1a_4-2a_2a_3)$.

13. $x^3-ax^2-x+a=0$. 14. x^2, y^2, z^2 are the roots of $x^3-3x+6=0$. 15. $\frac{18(a_1^2-a_0a_2)}{a_0^2}$. 16. $\frac{48(a_1^2-a_0a_2)}{a_0^2}$.

18. $x^3-\frac{6a_2}{a_0}x^2+\frac{4(4a_1a_3-a_0a_4)}{a_0^2}x-\frac{8}{a_0^3}(2a_0a_3^2-3a_0a_1a_4+2a_1^2a_4)=0$

20. $3p^3-16pq+64r$; $\frac{1}{r^4}(q^3-p^3r)$.

23. $3x^3-9x^2+6x+2=0$. 24. $2s^2-p$, $\frac{1}{3}(3q-6sp+4s^3)$.

Examples III (h) (P. 97)

1. $x^3-\frac{15}{2}x^2+\frac{225}{16}x-\frac{125}{16}=0$.

2. $z^3+6qz^2+9q^2z+27r^2+4q^3=0$. 3. $x^3+\frac{9}{2}x^2+\frac{81}{16}x+\frac{135}{16}=0$.

4. $x^3-\frac{98}{9}x^2+\frac{2401}{81}x=0$. The original equation has two equal roots.

Examples III (i) (Pp. 97–100)

1. 3, $2\pm2i$. 5. $y^3-2by^2+(ac+b^2)y+c^2-abc=0$.

8. $(y^2-ay+b-c)(y-b+c)-c(y+1-a)^2=0$.

9. The transformation is $x=\frac{d}{c-y}$.

10. $y^4+(2b-a^2)y^3+(b^2-2ac+2d)y^2+(2bd-c^2)y+d^2=0$.

11. $(5y^2+7y+1)^2=y(3y^2+y+2)^2$.

13. $8y^3-25y^2+44y-27=0$. 14. $y^4-8y^3+14y^2+8y+1=0$.

17. The transformation is $x^2=qy$, $y(qy+p)^2=q$.

18. $\pm 2, \pm 1, 5$. 19. $\pm 1, \pm 3, 4$.

20. $z^3+\frac{9Hz^2}{a_0^2}-\frac{27(G^2+4H^3)}{a_0^6}=0$.

Examples IV (*a*) (Pp. 104-105)

2. One positive and two complex.
4. One positive and four complex. 6. ± 1.
7. One positive and $2n$ complex. 8. $2n$ complex roots; -1.
9. Three positive and one negative.

Examples IV (*b*) (Pp. 108)

1. (1, 2). 3. $(-2, -1)$; $(-1, 0)$; $(0, 1)$; $(5, 6)$.
6. The new equation $9y^5-19y^4-57y^3-55y^2-10y-1=0$, has only one positive root. The original equation has one positive and four complex roots.

Examples IV (*c*) (Pp. 113-114)

2. $3, 3, -3, -\frac{1}{2}$. 3. $2, 2, -2, -2, -\frac{1}{2}$.
4. $(x^2-2x+3)^2$. 5. $2, 2, 2, -3$.
6. All the roots are equal, each being equal to 3.
7. 2 roots are equal to 2, and 4 roots equal to -1.
8. (*i*) 3, 3, -1, -1, -1; (*ii*) 1; 1, 1, $x^2+2x+3=0$.
11. (*ii*) $k=0$, or $\frac{-4}{27}$. The roots are $-1, -1, -2$.

or $\frac{-5}{3}, \frac{-5}{3}, \frac{-2}{3}$. $\frac{-4}{27} \leqslant k \leqslant 0$.

14. One negative and two positive.
15. The roots are all real, if $4 \leqslant k \leqslant 5$.
16. $a=2$ or $\frac{54}{25}$; when $a=2$, roots are 1, 1, $\frac{5}{2}$; and when $a=\frac{54}{25}$, the roots are $\frac{5}{3}, \frac{5}{3}, \frac{5}{6}$. 17. $a>16$, or $a<-16$.
19. $8 \leqslant k \leqslant 11$. 20. $0 \leqslant a \leqslant 5$. 23. $a=-1$, $x=-2$.

Examples IV (*d*) (Pp. 117-119)

2. Positive roots: 5 and $\frac{60}{73}$; negative roots: $-8, \frac{-146}{171}$.
6. 2 and -7. 7. $\frac{5}{2}$. 8. 5. 9. 3.

Examples IV (*e*) (Pp. 120-121)

1. $(-2, -1)$; (1, 2); (3, 4). 2. Two positive roots; one is 1, and the other lies between 2 and 3. The remaining roots are complex. 4. One negative root in the interval $(0, -1)$; two positive in the interval (1, 2). 5. (2, 3); (3, 4); $(-1, -2)$.
6. $(-2, -3)$.

Examples IV (*f*) (Pp. 125-127)

1. 3, −1. 4. ±1. 5. 4, −1, −2, −3. 6. 7, −2, −2, −2.

Examples IV (*g*) (Pp. 128-129)

2. 8, 8, $-\frac{1}{2}$, ±1. 4. −1, 3, 5, $\frac{2}{3}$. 5. $\frac{3}{2}$, $-1, \frac{-1\pm 3i}{2}$.

6. *For* $-218x$ *read* $-281x$; 5, 8, 11, $\frac{-1\pm\sqrt{-3}}{2}$.

7. $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, -6$. 8. $\frac{6}{9}, \frac{-4}{9}, \frac{3}{9}, 1\pm\sqrt{2}$.

10. 2, 5, −2, −7. 11. $\frac{1}{2}, \frac{2}{3}$. 12. $(x-2)(x+2)^2(x+1)(x^2-2x+4)$.

13. 6, 4, 1, −1, −5, −6; $x^2-x+1=0$. 14. 12, 35.

Examples IV (*h*) (Pp. (137-138)

1. 12·5 2. 125. 3. 3·05. 4. 3·2842.

5. 1·25992. 6. 1·44225. 7. −1·517. 8. 12·969. 9. 9·886.

Examples IV (*i*) (Pp. 140-141)

1. 1·51850605. Negative roots are −0·50849, −1·24359.

3. 3·128, 1, −1. 4. 1·216396757, −1·3300587395.

6. 2·24004. 7. 0·899607. 8. 2·0472755. 9. 2·9241769934.

10. 1·3036137, −7·4613430, each correct to 7 decimal places.

11. 4·581400362. 12. 2·0945514815.

13. 2·064434533. 14. 3·6933320241.

15. 1·7603017589, correct to ten decimal places.

Examples IV (*j*) (Pp. 142-143)

2. 1·35689584; 1·69202147. 3. 1·2556; 1·77733.

4. 23·2131277; 23·2295212. 5. 2·16, 2·41. 6. 4·242; 4·246.

7. 0·1127016; 0·8872983; 0·5.

8. 0·0694318; 0·3300094; 0·9305681.

Examples V (*b*) (Pp. 147-148)

3. (*i*) 1, 2, −3. (*ii*) +2, −1, −1. (*iii*) 3; the other two roots are complex.

Examples V (*c*) (Pp. 151-152)

3. 3, 3, −6; (*ii*) 3, $\frac{-3\pm i\sqrt{3}}{2}$; (*iii*) 5, $\frac{1}{2}(-5\pm 3\sqrt{-3})$;

(*iv*) $-\frac{1}{2}, \frac{1}{2}(-1\pm\sqrt{-3})$; (*v*) 17, 17, −19; (*vi*) $\frac{1}{3}, \frac{-7\pm 9\sqrt{-3}}{6}$;

(*vii*) −1, −1, −4. (*viii*) $2+u+v, 2+\omega u+\omega^2 v, 2+\omega^2 u+\omega v$, where $u^3=18, v^3=12$.

(*ix*) $-4,\ 5\pm\sqrt{-3}$. (*x*) $4x+3=\sqrt[3]{-135},+\sqrt[3]{25}$,
$\omega\sqrt[3]{-135}+\omega^2\sqrt[3]{25},\ \omega^2\sqrt[3]{-135}+\omega\sqrt[3]{25}$.

Or, the equation is equivalent to $5(x+3)^3+(3x+1)^3=0$.

(*xi*) The equation can be written as

$2(x+1)^3-(x-2)^3=0$. Or, $x+4=-3(\sqrt[3]{4}+\sqrt[3]{2})$, etc.

i.e. $-4-3\ (2^{\frac{2}{3}}+2^{\frac{1}{3}}),\ -4-3(2^{\frac{2}{3}}\omega+2^{\frac{1}{3}}\omega^2),\ -4-(2^{\frac{2}{3}}\omega^2+2^{\frac{1}{3}}\omega)$.

4. $q^3+8r^2=0$; $\frac{3}{2},\ \frac{-3\pm\sqrt{5}}{4}$.

5. The roots are real, if $4\leqslant k\leqslant 5$.

6. $-(a+1),\ -(\omega a+\omega^2)$. $-(\omega^2 a+\omega)$.

7. $-(a+b),\ -(\omega a+\omega^2 b),\ -(\omega^2 a+\omega b)$. 8. $\frac{-3a}{2},\ \frac{-3a}{2},\ 3a$.

Examples V (*d*) (Pp. 156)

1. $-6,\ 3\pm i$. 2. $-2,\ 1\pm 2i$. 3. $-2,\ 3,\ 6$. 4. $2,\ -3,\ 4$.

Examples V (*c*) (Pp. 161-165)

5. (*i*) $p^2=4$; $1\pm\sqrt{-2},\ -1\pm\sqrt{2}$. (*ii*) $p^2=4$; $4,\ -2,\ -1\pm\sqrt{-1}$.
(*iii*) $p^2=1$; $\frac{1}{2}(-1\pm\sqrt{-3}),\ \frac{1}{2}(1\pm\sqrt{21})$.
(*iv*) $p^2=4$; $-1\pm\sqrt{2},\ 1\pm 2i$. (*v*) $\pm\sqrt{3},\ 1\pm\sqrt{-6}$.
(*vi*) 1, 2, 3, 4. (*vii*) $-1\pm\sqrt{2},\ \frac{1}{2}(-1\pm\sqrt{13})$.
(*viii*) $2\pm\sqrt{3},\ 3\pm 2\sqrt{2}$. (*ix*) $\frac{1\pm i}{2},\ -2\pm\sqrt{2}$.

6. $(x^2+x\sqrt{6}+3-\sqrt{6})(x^2-x\sqrt{6}+3+\sqrt{6})$.

7. $2\pm 3\sqrt{2},\ -2\pm\sqrt{-2}$.

8. The equation takes the form $(x+2)^4+2(x-1)^4=0$.

Examples VII (*a*) (Pp. 189-190)

Cogt. and Digt. stand for 'Convergent' and 'Divergent,' srepectively; coges., and diges., for 'converges' and 'diverges'.

1. (*i*) Digt. (*ii*) cogt. (*iii*) oscillates finitely.
(*iv*) cogt. (*v*) $r>1$, digt.; $-1<r<1$, cogt.; $r=-1$, oscillates finitely between -1 and 1; $r<-1$, oscillates infinitely; $r=1$, reduces to a trivial case when each term of the sequence equals unity.
(*vi*) Oscillates infinitely between zero and ∞.

2. (*a*) The least integer greater than $\frac{\pm 1+\sqrt{1+4G^2}}{2}$; (*b*) 10.

Examples VII (*b*) (Pp. 191-192)

1. $\frac{17}{24}$. 2. Coges. to $\frac{41}{48}$. 3. If n be a positive ineger,
$s_{4n}=0,\ s_{4n+1}=13,\ s_{4n+2}=6,\ s_{4n+3}=2$.

4. Coges. to 1. 6. Oscillates infinitely.

Examples VII (*c*) (Pp. 199-203)

9. Coges. for $a>0$, diges. for $a\leqslant 0$. 10. Coges. if $p>1$; otherwise diges. 15. Coges. 16. Coges. 17. Diges.
18. Coges. 19. Diges. 25. Coges. if $p>2$; otherwise diges.
28. Coges. if $q>p+1$; diges. if $q\leqslant p+1$. 29. Digt.
31. Coges. if $k>4$; otherwise diges.

Examples VII (*d*) (Pp. 205—208)

4. Cogt. 5. Digt. 6. Cogt. 7. Cogt.
17. Coges. if $-1<x<1$; diges. if $|x|>1$. For $x=1$, coges. if $p>1$, and diges. if $p\leqslant 1$.
18. Coges. if $0<x<1$; diges. if $x\geqslant 1$.
19. Coges. if $x<3$; diges. if $x>3$, (*ii*) coges.

Examples VII (*e*) (Pp. 218—221)

2. Coges. if $x\leqslant 1$; diges. if $x>1$. 3. Diges.
4. Coges. if $a>2$; diges., if $a\leqslant 2$. 5. (*a*) Coges. if $x<1$; diges. if $x\geqslant 1$; (*b*) coges. if $x<1$; diges. if $x\geqslant 1$.
6. Digt. 7. Coges. if $x\leqslant 1$, and diges. if $x>1$.
14. Coges. if $ex<1$; diges. if $ex\geqslant 1$.

Examples VII (*f*) (Pp. 224—226)

7. Coges. absolutely for finite values of x. 10. Cogt. 11. Digt.

Examples VII (*h*) (Pp. 233—235)

1. Digt. 2. Coges. for $-1<x<1$, diges. for $|x|>1$, or $x=1$. 3. Coges. if $x^2\leqslant 1$, diges. for $x^2>1$.
4. Coges. if $-1<\frac{bx}{a}<1$; diges. if $\frac{bx}{a}>1$; oscillates finitely or infinitely, according as $\frac{bx}{a}=-1$, or <-1.
7. Coges. 8. Coges. if $|x|<1$, diges. if $|x|\geqslant 1$.
14. Coges. for $|x|<1$, diges. for $x\geqslant 1$; for $x=-1$, converges conditionally. 15. Coges. for $x<1$, diges. for $x\geqslant 1$.
16. Coges. if $|x|<1$, diges. if $x\geqslant 1$; conditionally coges. if $x=-1$.

Examples VIII (Pp. 243—245)

4. 0. 5. 1. 6. ∞. 23. $\pm\sqrt{\frac{2}{5+2\sqrt{2}}}$.
24. $x=\pm 13$. 25. $1, -\frac{1}{6}$. 26. $\frac{a+b}{1-ab}$. 30. $x=\frac{1}{8}$.

Examples IX (*a*), (Pp. 254-255).

1. $2 \operatorname{cis} \frac{\pi}{3}$. 2. $\sqrt{2} \operatorname{cis} \frac{\pi}{4}$. 3. $2 \operatorname{cis} \frac{5\pi}{6}$.

4. $\sqrt{2(2+\sqrt{2})} \operatorname{cis} \tan^{-1} \frac{1}{1+\sqrt{2}}$.

5. $\sqrt{5(7+2\sqrt{5})} \operatorname{cis} \tan^{-1} \frac{1}{\sqrt{5}+1}$.

6. $\sec \alpha \operatorname{cis} \alpha$, or $(-\sec \alpha) \operatorname{cis} (\pi+\alpha)$.

7. $\operatorname{cosec} \alpha \operatorname{cis} \left(\frac{\pi}{2}-\alpha\right)$, or $-\operatorname{cosec} \alpha \operatorname{cis} \left(\frac{3\pi}{2}-\alpha\right)$.

8. $\sec \alpha \operatorname{cis} \left(-\frac{\pi}{2}+\alpha\right)$, or $(-\sec \alpha) \operatorname{cis} \left(\frac{\pi}{2}+\alpha\right)$.

Examples IX (*b*), (Pp. 259-260).

1. $2i$, $-2i$, -1, $\frac{a^2-b^2}{a^2+b^2}+\frac{2iab}{a^2+b^2}$,
$\frac{a^4-6a^2b^2+b^4}{(a^2+b^2)^2}+4i\,\frac{a^3b-ab^3}{(a^2+b^2)^2}$, $-\frac{2}{25}+\frac{11i}{25}$, $1-i$.

2. $\operatorname{cis}(\alpha+\beta+\gamma)$, $2\cos\theta \operatorname{cis}(-3\theta)$, $i\tan\theta$.

3. $1, \frac{\pi}{3}$; $1, \frac{\pi}{2}$. 5. $\sqrt{2}, \frac{-\pi}{4}$; $\sqrt{5}, \tan^{-1}\left(-\frac{1}{2}\right)$.

Examples IX (*c*), (Pp. 261-265).

1. $(x^2-y^2)+2ixy$, $x^3-3xy^2+i(3x^2y-y^3)$,
$\frac{(ax+b)(cx+d)+acy^2}{(cx+d)^2+c^2y^2}+\frac{i\,(ad-bc)y}{(cx+d)^2+c^2y^2}$.

4. $\frac{\pi}{2}$, $\sqrt{13}$. The parallelogram is found to be a square.

8. $-6+9i$. 24. $am(a-b)=am(c-d)$.

Examples X (*a*) (Pp. 268—269)

1. $-2, -\frac{1}{2}, \pm i$. 2. $1, 2, \frac{1}{2}, -3, -\frac{1}{3}$,
3. $\pm 1, 2, \frac{1}{2}, -3, -\frac{1}{3}$. 4. $\pm 1, 2, \frac{1}{2}, \frac{1\pm\sqrt{-3}}{2}, \frac{-3\pm\sqrt{5}}{2}$.

5. $-1, -1, \frac{1}{2}(-1\pm i\sqrt{3})$.
6. $1, -\frac{1}{2}(1\pm\sqrt{-3}), \frac{1}{2}(7\pm\sqrt{45})$.

Examples X (*c*) (Pp. 272-276).

1. $\cos 10\theta - i \sin 10\theta$. 2. 1. 3. 1.

4. $2^8 \sin^8 \frac{\theta}{2}(\cos 4\theta - i \sin 4\theta)$.

5. $2^{10}\cos^{10}\frac{\theta}{2}(\cos 5\theta+i\sin 5\theta)$. 6. $\frac{n\pi}{3}$.

7. $\frac{4n\pi}{p(p+1)}$. 13. $2^{1/6}\operatorname{cis}\left(\frac{2n\pi}{3}+\frac{7\pi}{12}\right)$, $n=0, 1, 2$.

15. $\operatorname{cis}\left(\frac{n\pi}{2}+\frac{\pi}{6}\right)$, where $n=4k, 4k+1, 4k+2, 4k+3$; k being an integer. This gives $-\frac{1}{2}+i\frac{\sqrt{3}}{2}$, $-\frac{\sqrt{3}}{2}-\frac{i}{2}$, $\frac{1}{2}-i\frac{\sqrt{3}}{2}$, $\frac{\sqrt{3}}{2}+\frac{i}{2}$. 21. $\cos\frac{2a+2\pi r}{3}+i\sin\frac{2a+2\pi r}{3}$ $(r=0, 1, 2)$.

22. $\cos 3a+i\sin 3a$.

Examples X (*d*), (P. 278).

Corrigenda : For Ex. (*c*) (P. 278) *read* Ex. (*d*)
and for Ex. (*d*) (P. 282), *read* Ex. (*e*).

2. $\frac{7t-35t^3+21t^5-t^7}{1-21t^2+35t^4-7t^6}$, where $t=\tan\theta$.

Examples X (*e*) (Pp. 282—283)

1. $\sqrt[4]{5}\operatorname{cis}\frac{2n\pi+a}{4}$, $n=0, 1, 2, 3$, $a=\tan^{-1}\frac{4}{3}$;

$x=\sqrt[4]{5}\cos\frac{2n\pi+a}{4}$, $y=\sqrt[4]{5}\sin\frac{2n\pi+a}{4}$, $n=0, 1, 2, 3$.

3. 1. 4. $(-1)^q$. 17. 1

Examples XI (*b*), (Pp. 294-298).

1. -1 and $\cos\frac{p\pi}{7}\pm i\sin\frac{p\pi}{7}$, $p=1, 3, 5$.

2. $-1, \pm\operatorname{cis}\frac{\pi}{3}, \pm\operatorname{cis}\frac{\pi}{4}, \pm\operatorname{cis}\frac{3\pi}{4}$.

3. $\operatorname{cis}\frac{(2p+1)\pi}{5}$, $p=0, 1, 3, 5$.

Examples XII (*c*). (Pp. 316-318)

9. $\cos x\cosh y$, $-\sin x\sinh y$.

Examples XII (*d*). (Pp. 319-321).

2. (*i*) $x=a\sinh\theta$. (*ii*) $x=a\cosh\theta$. (*iii*) $x+1=\sqrt{2}\sinh\theta$. (*iv*) $x+\frac{3}{4}=\frac{\sqrt{23}}{4}\sinh\theta$. 13. $v=\frac{1}{2}\tanh^{-1}\frac{2y}{1+x^2+y^2}$.

Examples XIII (*a*) (P. 326)

1. $\sin\frac{nx}{2}\sin\frac{n+1}{2}x \;\Big|\; \sin\frac{x}{2}$. 2. $\sin\frac{nx}{2}\cos\frac{n+1}{2}x \;\Big|\; \sin\frac{x}{2}$

3. $\sin^2 nx/\sin x$. 4. $\sin 2nx/\sin 2x$. 5. 0.

6. $\sin\left(x-\frac{n-1}{2}y\right)\sin\frac{ny}{2} \;\Big|\; \sin\frac{y}{2}$.

7. $\sin\frac{n(\pi-x)}{2}\sin\frac{(n+1)(\pi-x)}{2} \;\Big|\; \cos\frac{x}{2}$.

8. $\sin\frac{n}{2}\left(\frac{\pi}{2}-x\right)\sin\frac{n+1}{2}\left(\frac{\pi}{2}-x\right) \;\Big|\; \sin\left(\frac{\pi}{4}-\frac{x}{2}\right)$.

9. $\sin\frac{n}{2}\left(\frac{\pi}{2}+x\right)\sin\frac{n+1}{2}\left(\frac{\pi}{2}+x\right) \;\Big|\; \sin\left(\frac{\pi}{4}+\frac{x}{2}\right)$.

10. $\sin\frac{n}{2}\left(\frac{\pi}{2}-x\right)\cos\frac{n+1}{2}\left(\frac{\pi}{2}-x\right) \;\Big|\; \sin\frac{1}{2}\left(\frac{\pi}{2}-x\right)$.

11. $\sin\frac{n}{2}(y+\pi)\cos\left(x+\overline{n-1}\,\frac{y+\pi}{2}\right) \;\Big|\; \cos\frac{y}{2}$.

12. $\sin(n+1)x\sin nx/2\sin x$.

13. $\frac{n}{2}-\frac{1}{2}\sin nx\cos(n+1)x/\sin x$.

14. $\frac{n}{2}\cos(n+1)x\sin nx/2\sin x$.

15. $$\frac{3\sin(n+1)\frac{x}{2}\sin\frac{nx}{2}}{4\sin\frac{x}{2}}-\frac{\sin\frac{3}{2}(n+1)x\,.\,\sin\frac{3nx}{2}}{4\sin\frac{3x}{2}}.$$

16. $$\frac{3}{4}\sin\frac{nx}{2}\cos\frac{n+1}{2}x \;\Big|\; \sin\frac{x}{2}$$
$$+\frac{1}{4}\sin\frac{3nx}{2}\cos\frac{3(n+1)x}{2} \;\Big|\; \sin\frac{3x}{2}.$$

17. $\frac{n}{2}\sinh 4nx \;\Big|\; 4\sinh 2x$.

18. $\sin\frac{pn\pi}{2n+1}\cos\frac{p(n+1)\pi}{2n+1} \;\Big|\; \sin\frac{p\pi}{2n+1}$, $p=(2n+1)k$.

Examples XIII (*b*). (Pp. 329-330)

1. $n\cos\frac{x}{2}-\frac{1}{2}\sin nx\cos(n+2)x/\sin x$.
2. $\frac{1}{2}(\cos 2x-\cos 2^{n+1}x)$.

3. $\frac{1}{2}\frac{\sinh 2nx}{\sinh 2x}\sinh(2n+1)x-\frac{n}{2}\sinh x$. 4. $\cot\frac{a}{2n}-\cot a$.

5. $\cot a-2^n\cot 2^n a$. 6. $\cot a\tan(n+1)a-(n+1)$.

7. $\tan 2^n a-\tan a$. 8. $\sin 2na/\sin a\sin(2n+1)a$.

9. $\{\tan(2n+1)a-\tan a\}/\sin 2a$.

10. $\sin na/\sin a\cos a\cos(n+1)a$.

11. $\{\sec(2n+1)a-\sec a\}/2\sin a$.

12. $[\operatorname{cosec} a-\operatorname{cosec}(2n+1)a]/2\sin a$.

13. $\frac{1}{4}\left[3\sin a-\frac{1}{3^{n-1}}\sin 3^n a\right]$. 14. $\frac{1}{2}(\cot a-\cot 3^n a)$.

15. $\cot\theta(\cot\theta-\cot\overline{n+1}\,\theta)-n$.

19. $\operatorname{cosec}\beta\{\cot a-\cot(a+n\beta)\}$.

20. $\frac{1}{2^{n-1}}\tan 2^n\theta-2\tan\theta$. 21. $\tan^{-1}\frac{n}{n+2}$.

For "$\tan^{-1}\frac{1}{1+1\cdot 2}+$" *read* "$\tan^{-1}\frac{1}{1+1\cdot 2}=$"

Examples XIII (*c*). (P. 331)

1. $\frac{2-2\cos x+\sin x+(n+1)\sin(n-1)x-n\sin nx}{2(1-\cos x)}$.

2. $\frac{\cos nx+2n\sin\frac{2n+1}{2}x\sin\frac{x}{2}-1}{2(1-\cos x)}$.

3. $\frac{(n+1)\sin nx-n\sin(n+1)x}{2(1-\cos x)}$.

4. $\frac{\sin x+(2n+3)\sin nx+(2n+1)\sin(n+1)x}{2(1-\cos x)}$.

Examples XIII (*d*). (Pp. 336-338)

1. $\frac{n}{2}\cos a+\frac{\cos(2\theta+na)\sin na}{2\sin a}$.

2. $\frac{n}{2}\sin p\theta+\frac{\sin(p+n+1)\theta\sin n\theta}{2\sin\theta}$.

3. $\frac{-\cos\theta+(n+1)\cos(n-1)\theta-n\cos n\theta}{2(1-\cos\theta)}$.

4. $3\cot\frac{a}{2}-\frac{\sin(a+na/2)\sin(n-1)a/2}{2(\sin^3 a/2)}$

$$+\frac{(n+2)(n+5)\sin na-(n+2)(n+3)\sin(n+1)a}{4(\sin^2 a/2)}.$$

5. If S is the sum, $2(1-\cos\theta)\,S=\dfrac{-2\sin n\theta/2}{\sin\theta/2}\cos\dfrac{n-1}{2}\theta$
$+n(n+3)\cos n\theta-n(n+1)\cos(n+1)\theta.$

10. (*i*) $\cot\theta\cos^n\theta\sin\theta$, (*ii*) $\cot\theta\,(1-\cos^n\theta\cos n\theta)$.

11. (*i*) $\cot\theta-\dfrac{\cos(n+1)\theta}{\sin\theta\cos^n\theta}$, (*ii*) $\dfrac{\sin(n+1)\theta}{\sin\theta\cos^n\theta}-1$.

12. $2^n\cos^n\dfrac{\theta}{2}\cos\dfrac{n\theta}{2}$.

13. $\tan x-\tan x/2^n$.

14. $2^{n-2}\sin\theta/2^{n-1}-\frac{1}{4}\sin 2\theta$.

15. $e^{\cos^2\theta}\cos\left(\theta+\dfrac{\sin 2\theta}{2}\right)$.

16. $e^{\sin 2\theta/2}\cos(\theta+\sin^2\theta)$.

17. $\dfrac{\cos\theta}{1-\cos\theta}-\log(1-\cos\theta)$.

18. $\log\dfrac{1+xe^{i\theta}}{1+xe^{-i\theta}}$.

19. $\log 2\cos\dfrac{\theta}{2}$.

20. $e^{-\cos\theta}\sin(\sin\theta)$.

21. $\cosh(\sin\theta)\cos(\cos\theta)$.

22. $\dfrac{4\sin\theta}{5-4\cos\theta}$. 23. $\dfrac{4\cos\theta-2}{5-4\cos\theta}$. 24. $\dfrac{1-x\cosh\alpha}{1-2x\cosh\alpha+x^2}$.

25. $\dfrac{x\sinh\alpha}{1-2x\cosh\alpha+x^2}$.

26. $e^{x\sin\theta}\sin(x\sin\theta)$.

27. $e^{-x\cos\theta}\cos x\sin\theta$.

28. $e^{x\cos\beta}\sin(\alpha+x\sin\beta)$.

29. $e^{x\cos\beta}\cosh\alpha+x\sin\beta$.

30. $(2\sin\alpha)^{-\frac{1}{2}}\sin\left(\dfrac{\pi}{4}+\dfrac{\alpha}{2}\right)$,
except when $\alpha=n\pi$.

31. $-1+\dfrac{1}{(2\sin\alpha/2)^n}\cos\dfrac{n}{2}(\pi-\alpha)$, where $n<1$ and $2\pi>\alpha>0$.

32. $\sqrt{2\cos\alpha/2}\sin\alpha/4$, when $-\pi<\alpha<\pi$.

33. $e^{-\cos\alpha\cos\beta}(\cosh\sin\beta)$.

34. $e^{\cosh u}\sinh(\sinh u)$.

35. $\dfrac{x}{\sqrt{1-x^2\sin 2\alpha+\alpha^2/4}}\sin(\alpha+\dfrac{\theta}{2})$,
where $\tan\theta=\dfrac{x^2\sin 2\alpha}{2-x^2\cos 2\alpha}$, and when $\sqrt{2}>x>-\sqrt{2}$.

36. (*i*) $\sin(x\sin\theta)\sinh(x\cos\theta)$,
(*ii*) $\cos(x\sin\theta)\cosh(x\cos\theta)$.

37. $\frac{1}{2}\log\tan\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right)$.

38. $\frac{1}{4}\log\dfrac{1+2c\cos\alpha+c^2}{1-2c\cos\alpha+c^2}$.

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